Capacitors and Dielectrics

The ideas of energy storage in E-fields can be carried a step further by understanding the concept of "Capacitance."

Consider a sphere with a total charge, Q, and a radius, R. From previous problems we know that the potential at the surface is, $V = k\dfrac{Q}{R}$

Putting more charge on the sphere stores more energy, but the ratio of energy or potential to charge depends only on R, not on Q or V. That is,

$$\frac{Q}{V} = \frac{R}{k} = 4\pi\varepsilon_0 R.$$

It's true for all charged objects that the ratio of potential to voltage depends only on the shape, so this ratio is defined as the capacitance.

$$\boxed{C \equiv \frac{Q}{V}}$$

.

The units of capacitance are $\dfrac{1\text{ coulomb}}{\text{volts}} \equiv 1\text{ Farad} \equiv\ 1\text{ F}$. Common values of capacitance are microfarads, μF (10^{-6} Farads) and picofarads, pF (10^{-12} Farads).

Consider two conductors connected to the terminals of a battery. The battery will supply an equal amount of charge, but of opposite sign, to each of the conductors. The question arising at this point: what will be capacitance of the conductor system? Let us consider different conductor systems:

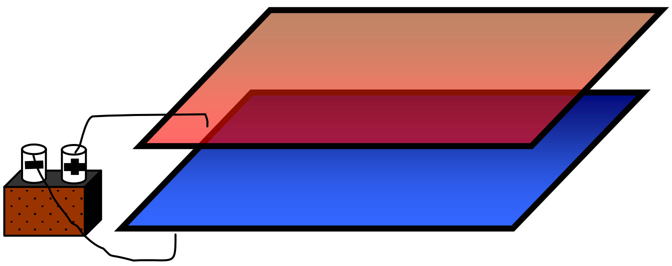
Parallel plates

Two conducting parallel plates separated by a distance d with charges +*Q* and -*Q*. The potential difference between the plates (from one plate to the other) is

$$V_a - V_b = V = Ed = \left(\frac{\rho_s}{\varepsilon_o}\right)d = \frac{Qd}{A\varepsilon_o}.$$

The capacitance is

$$\boxed{C = \frac{Q}{V} = \frac{\varepsilon_0 A}{d}}$$



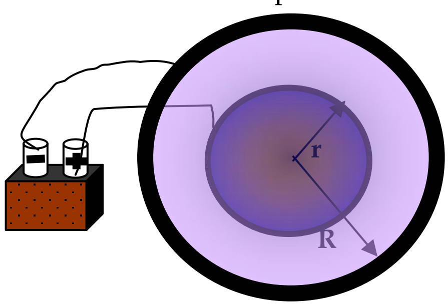
Conducting concentric spheres

Two concentric spheres of radii *R* and *r*. The potential difference between the spheres is

$$V_a - V_b = V = \frac{Q}{4\pi\varepsilon_o}\left(\frac{1}{R} - \frac{1}{r}\right).$$

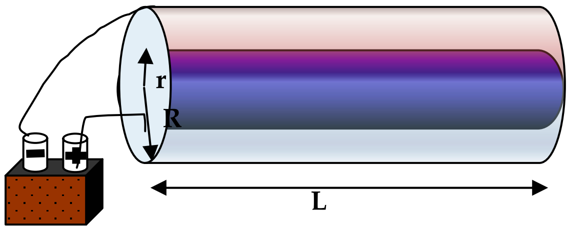
The capacitance is

$$\boxed{C = \frac{Q}{V} = \frac{4\pi\varepsilon_o}{\left(\frac{1}{R} - \frac{1}{r}\right)}}$$



Coaxial Cable

Coaxial cable (two concentric conducting cylinders) of length L. The inside conductor has a radius r with charge ρ_ℓ and the inside surface of the outside conductor is R with charge - ρ_ℓ.



$$V_a - V_b = V = \frac{\lambda}{2\pi\varepsilon_o}\ln\frac{R}{r} = \frac{Q}{2\pi\varepsilon_o L}\ln\frac{R}{r}$$

The capacitance is

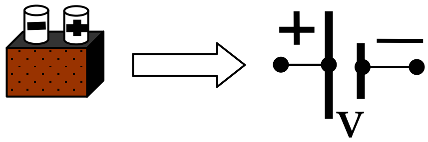
$$\boxed{C = \frac{Q}{V} = \frac{2\pi\varepsilon_o L}{\ln\frac{R}{r}}}$$

Exercise

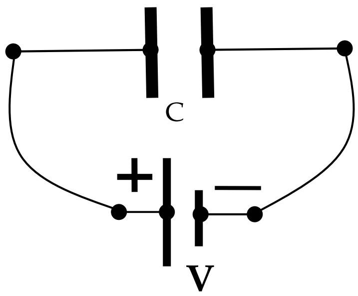
There are electrical devices that are designed to store energy in this fashion. These devices are referred to a "capacitors." To get an idea of the magnitude of the unit Farad, find how large a parallel plate capacitor must be in order to have a capacitance of one Farad. Take the distance between the plates to be 0.1 mm.

Capacitors in Electrical Circuits

The circuit diagram of a capacitor

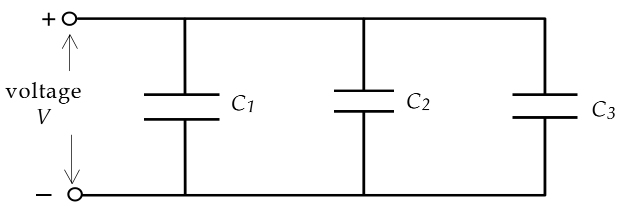


You can "charge" a capacitor by connecting the capacitor to a battery (power supply). (Remember that in the electrostatic situation the wires (conductors) are equipotentials.)



Combinations of Capacitors - this is necessary because capacitors with only certain values are available.

Capacitors in parallel: The total capacitance of the circuit C_{eq}, that is equivalent to the capacitors is parallel (does the same job as the capacitors in parallel).
"top to top, bottom to bottom"
"left to left, right to right"
"The voltage is the same across all capacitors in parallel." And charge is conserved:

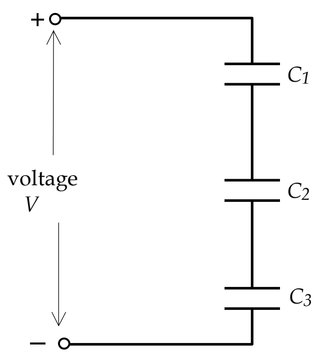


$V = V_{C_1} = V_{C_2} = V_{C_3}$; $Q = Q_1 + Q_2 + Q_3$

Using the definition of the capacitance:

$$Q = C_{eq}V = C_1V_1 + C_2V_2 + C_3V_3 = V\left(C_1 + C_2 + C_3\right)$$

$$C_{eq} = C_1 + C_2 + C_3$$



Capacitors in Series: In this case the capacitors connected to each others "one after another" - similar to a train engine pulling its cars. The total capabitance , C_{eq} can be obtained as follows.
"The <u>charge</u> on the capacitors that are in series is the same on each capacitor."

$V = V_{C_1} + V_{C_2} + V_{C_3}; \quad Q = Q_1 = Q_2 = Q_3$

Using the definition of the capacitance:

$$V = \frac{Q_1}{C_1} + \frac{Q_2}{C_2} + \frac{Q_3}{C_3} = Q\left(\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}\right)$$

$$C_{eq} = \frac{V}{Q} = \left(\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}\right)$$

We can generalize our results for the total capacitances in the parallel and series circuits:

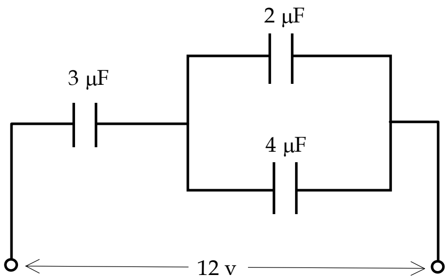
$$\boxed{\begin{array}{ll} C_{eq} = C_1 + C_2 + \cdots + C_N & \text{N-Capacitor connected in parallel} \\ \dfrac{1}{C_{eq}} = \dfrac{1}{C_1} + \dfrac{1}{C_2} + \cdots + \dfrac{1}{C_N} & \text{N-Capacitor connected in series} \end{array}}$$

For two capacitors connected in series: $C_{eq} = \dfrac{C_1 C_2}{C_1 + C_2}$.

Parallel and series combinations:

Example

Find the charge on each capacitor and the voltage across each capacitor.



Solution

The equivalent capacitance of the circuit is

$C_{eq} = (3) Series (2 // 4)$

$= 3 series (2 + 4) = \dfrac{3*6}{3+6} = 2\mu F$

The total charge $Q = VC_{eq} = 12*2 = 24\mu$ *Coulomb.*

And the charge on the $3\mu F$ capacitor is equal to the total charge: 24μ *Coulomb.* Potential(voltage) of this capacitor is $V = 24/3 = 8$ *Volts.*

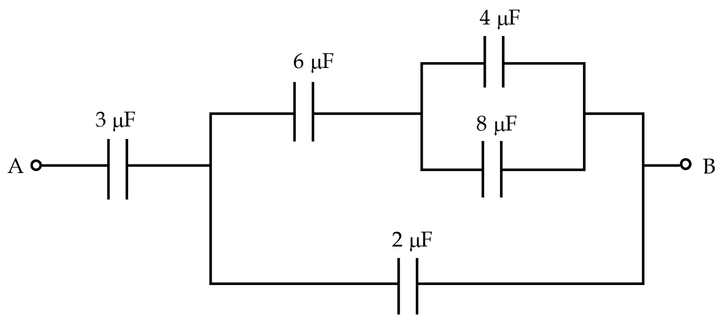
From the conservation of energy voltages of the 2 and $4\mu F$ capacitors are $V = 12 - 8 = 4$ volts.

Then the charge on the $2\mu F$ capacitor is $4*2 = 8\mu$ *Coulomb*

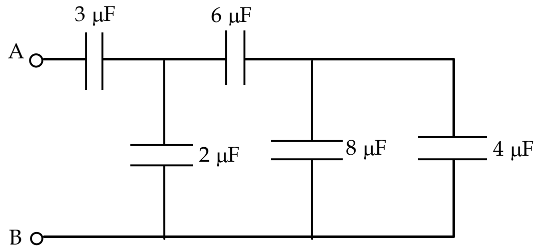
Then the charge on the $2\mu F$ capacitor is $4*2 = 16\mu$ *Coulomb.*

Exercises

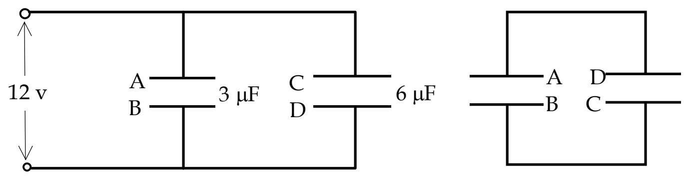
Find the equivalent capacitance between points A and B.



Find the equivalent capacitance between points A and B.



A 3 μF and a 6 μF capacitor are connected in parallel and are charged by a 12 volt battery, as shown. After the capacitors are charged, the battery is then disconnected from the circuit. The capacitors are then disconnected from each other and reconnected after the 6 μF capacitor is inverted. Find the charge on each capacitor and the voltage across each.



Energy stored in the capacitor.

When a capacitor is being "charged" by a battery (or power supply), work is done by the battery to move charge from one plate of the capacitor to the other plate. As the capacitor is being charged, we can say that the capacitor is storing energy (What kind of energy?). Find the stored energy.

Consider a capacitor being charged by a battery. After a time t, the voltage across the capacitor is V and an amount of charge q has accumulated (so far) on the plates of the capacitor. To move an additional amount of charge dq from one plate to the other, the battery must do an amount of work dW, where $dW = (dq)V$. (Remember from before that $W_{a\to b} = U_a - U_b = q(V_a - V_b)$, or $W = qV$ is the work done moving a charge q through a voltage V.) Formally:

$$dW = dU = Vdq = \frac{q}{C}dq \Rightarrow U = \int_0^Q \frac{q}{C}dq = \frac{Q^2}{2C}$$

where U is the stored energy in the capacitor. As a summary:

$$\boxed{U = \frac{Q^2}{2C} = \frac{1}{2}CV^2 = \frac{1}{2}QV}$$

Question: Where is the energy stored?

Energy density is defined as the stored energy per unit volume: $u = \dfrac{U}{volume}$

Example

Calculate the stored energy in a parallel plate capacitor of surface area A and plate separation d. Potential difference between the plates is V_0. Calculate energy density.

Solution: Capacitance of the parallel plate is: $C = \dfrac{Q}{V} = \dfrac{\varepsilon_0 A}{d}$. Then the energy is:

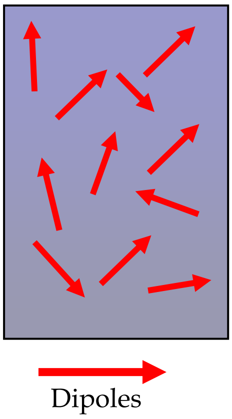
$$U = \frac{1}{2}\frac{\varepsilon_0 A}{d}V_0^2$$

In order to find energy density we divide U by volume $V=Ad$;

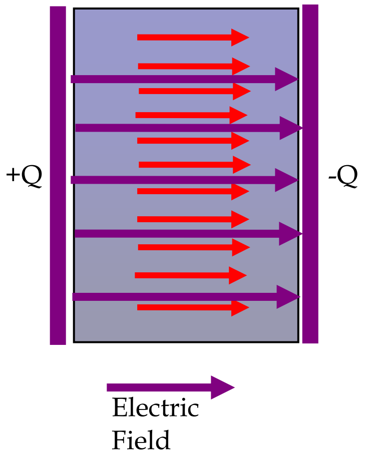
$$u = \frac{U}{Ad} = \frac{1}{2}\frac{\varepsilon_0}{d^2}V_0^2 = \frac{1}{2}\varepsilon_0 E^2$$

where E is the electric field between the plates and it can be defined as *V=E.d.*

Dielectrics in capacitors



A careful glance at the equations of the capacitors shows that we can increase the capacitance of a capacitor by using some materials whose permittivity bigger than the permittivity of the air ε_0. These materials are known as the dielectric materials. Dielectrics are insulators. Electrons are not free to flow from one molecule to another. The atoms in a dielectric can have **dipole** moments. In a typical chunk of dielectric material these dipoles are randomly aligned and therefore produce no net field as shown.



When a dielectric is placed between the plates of a capacitor with a surface charge density ρ_s the resulting electric field, E_0, tends to align the dipoles with the field. These results in a net charge density ρ_s induced on the surfaces of the dielectric which in turns creates an induced electric field, E_i, in the opposite direction to the applied field. The total field inside the dielectric is reduced to,

$$E = E_0 - E_i$$

The dielectric constant is defined as the ratio of the applied field to the total field, $\kappa = \frac{E_0}{E}$ (kappa). Substituting for E and solving for the induced field: $\kappa = \frac{E_0}{E_0 - E_i} \Rightarrow E_i = \left(1 - \frac{1}{\kappa}\right)E_0$.

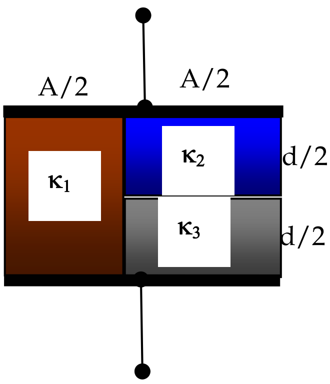
Note that κ=1 is a perfect insulator such as a vacuum and κ=∞ is a perfect conductor. How does the introduction of a dielectric affect the capacitance of a capacitor? We can find change in the potential:

$$V = \int \vec{E}.d\ell = \int \frac{1}{\kappa}\vec{E}_0.d\ell = \frac{1}{\kappa}V_0$$

If the capacitance without dielectric $C_0=Q/V_0$, with dielectric it will be C=Q/V, eliminating Q, V and V_0 between equations we obtain

$$\boxed{C = \kappa C_0}$$

The capacitance larger by a factor κ. Some values -- vacuum: κ = 1, glass: κ = 5 to 10, mica: κ = 3 to 6.



Example

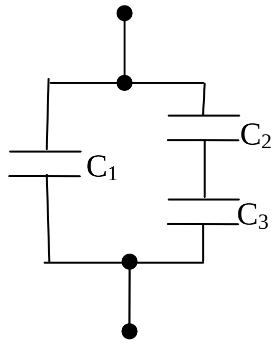
Find the capacitance of the capacitor shown in figure.

Solution

Draw the circuit diagram of the capacitors

These are parallel plate capacitors and their capavitance can be calculated by using $C_0 = \frac{\varepsilon_0 A}{d}$; $C = \kappa C_0$. Then:

$$C_1 = \kappa_1 \frac{\varepsilon_0 A/2}{d};\quad C_2 = \kappa_2 \frac{\varepsilon_0 A/2}{d/2};\quad C_3 = \kappa_3 \frac{\varepsilon_0 A/2}{d/2}$$



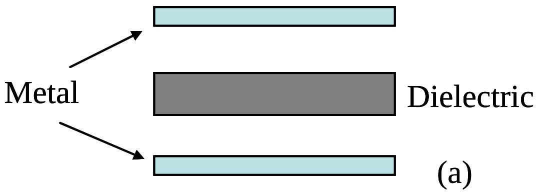
Since C_2 and C_3 are series to the each other, the equivalent capacitance is: $C_{eq1} = \frac{C_2 C_3}{C_2 + C_3} = \frac{\kappa_2 \kappa_3}{\kappa_2 + \kappa_3}\frac{\varepsilon_0 A}{d}$

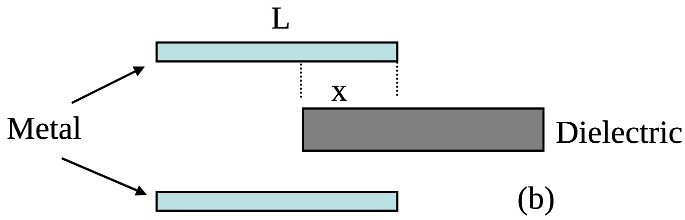
C_1 is parallel to the C_{eq1}:

$$C_{eq} = C_{eq1} + C_1 = \frac{\kappa_2 \kappa_3}{\kappa_2 + \kappa_3}\frac{\varepsilon_0 A}{d} + \kappa_1 \frac{\varepsilon_0 A}{2d} = \frac{\varepsilon_0 A}{d}\left(\frac{\kappa_2 \kappa_3}{\kappa_2 + \kappa_3} + \frac{\kappa_1}{2}\right).$$

Example

Consider a parallel capacitor made of two large metal plates of L by L separated by distance d (<<A) with a neutral dielectric slab (thickness a, same area as the metal plates). The potential difference between the two plates is V. Find the amount of charge on the plates and energy stored in (a) and (b).





Solution:

(a) We can think that two capacitor connected series with the capacitances

$$C_1 = \frac{\varepsilon_0 A}{d-a} = \frac{\varepsilon_0 L^2}{d-a};\ \text{and}\ C_2 = \frac{\kappa \varepsilon_0 L^2}{a}$$

The equivalent capacitance is

$$C_{eq} = \frac{C_1 C_2}{C_1 + C_2} = \frac{\kappa \varepsilon_0 L^2}{a + (d-a)\kappa}$$

The total charge on the plates $Q=CV$. Energy stored in the capacitor is: $W=QV/2$.

(b) Equivalent circuit of the configuration is a capacitor connected in parallel to the two capacitoe in series with the capacitances:

$$C_1 = \frac{\varepsilon_0 L(L-x)}{d};\quad C_2 = \frac{\varepsilon_0 Lx}{d-a};\quad C_3 = \frac{\kappa \varepsilon_0 Lx}{a}$$

The equivalent capacitance is

$$C_{eq} = C_1 + \frac{C_2 C_3}{C_2 + C_3};\ \text{Q=CV and W=QV/2.}$$

Exercises

A parallel plate capacitor consists of plates of area 10 cm^2 and a distance between the plates of 0.05 mm. The space between the plates is filled with a dielectric of constant κ = 5. The capacitor is connected to a 6 volt battery.

a. Find the capacitance of the capacitor with the dielectric.
b. Find the charge on the plates of the capacitor.
c. Find the induced charge on the surface of the dielectric.
d. Find the energy stored in the capacitor.
e. Find the energy density between the plates of the capacitor.

Find the capacitance of the capacitor shown.