

Chapter 4: Electric Potential

Introduction

Electric Potential or Potential or Voltage is an important concept related to the electric field and to energy. In order to understand potential concept let us summarize gravitational potential energy concept.

Similarity between the gravitational force and the electrical force.

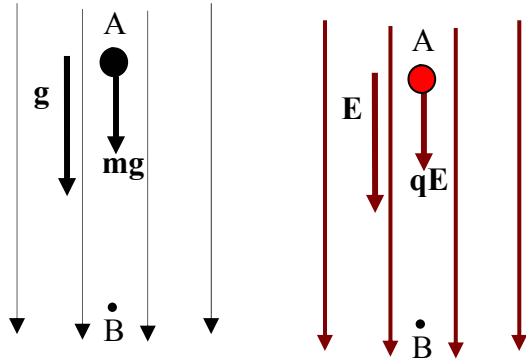
We know that the gravitational force is a conservative force, and we know that there is a *gravitational potential energy* associated with a *mass* placed in the gravitational field. And the change in the potential energy equals the negative of the work done by the conservative gravitational force. That is,

$$\Delta U = U_B - U_A = -W_{A \rightarrow B}$$

or,

$$W_{A \rightarrow B} = U_A - U_B$$

In words, the work done by the conservative force moving the object from A to B equals the object's initial potential energy minus the final potential energy. At the surface of the earth, the gravitational field points downward (the direction the gravitational force acts on objects).



Suppose an object of mass m is moved from point A to point B in the gravitational field. Is the work done by gravity, $W_{A \rightarrow B}$, positive or negative? And which point, A or B, represents a higher potential energy?

Note that the equation that describes the electrical force has the same form as the gravitational force:

$$F = G \frac{m_1 m_2}{r^2} \quad \text{and} \quad F = k \frac{q_1 q_2}{r^2}, \quad \text{and}$$

$$F = mg = \text{constant} \quad \text{and} \quad F = qE = \text{constant}.$$

Because the forces are analogous, we can conclude that the electrical force is also a conservative force and gives rise to a potential energy. This is called the *electrical potential energy* associated with a *charge* in an electric field.

As we mentioned in the first chapter the difference between gravitational forces and energies and electrical forces and energies is that the *mass* in gravity is always *positive*, whereas the charge in *electricity* can be *positive* or *negative*.

For a more general analysis, consider a charge q moving from point A to point B in the electric field shown. The work done by the electric field as the charge moves from A to B is written as

$$W_{A \rightarrow B} = -\Delta U = -(U_B - U_A) = U_A - U_B,$$

or,

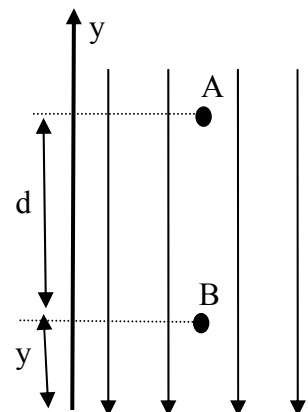
$$U_A - U_B = W_{A \rightarrow B} = \int_A^B \vec{F} \cdot d\vec{l} = q \int_A^B \vec{E} \cdot d\vec{l}$$

$$\Delta U = -q \int \vec{E} \cdot d\vec{l}.$$

This is the electric potential energy.

Exercises

1. A uniform electric field \vec{E} points directly downward as in figure.

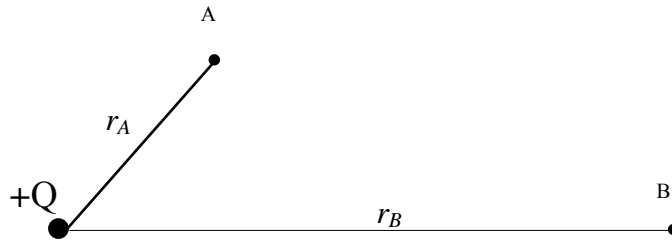


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- What is the difference in the potential energies of a charge $+q$ placed in positions A and B in the E-field?
- If the charge $+q$ is released from rest at point A, at what velocity will it be traveling as it passes point B?

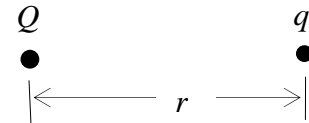
2. A point charge $+Q$ creates an electric field in space.

- Find the difference in the potential energy ($U_A - U_B$) of a charge $+q$ between the positions A and B that are a distances r_A and r_B from charge the charge $+Q$.



- At what point does the charge have a higher potential energy?
- If the charge $+q$ is released from rest at point A, then how fast is it traveling when it reaches infinity?
- Answer question (a) if the charge is $-q$.
- Answer question (a) if the charge $+Q$ is $-Q$, and $q = +q$.

3. How much work must *you* do in order to move a charge q from infinity to a distance r from a charge Q ?



Example

Find the change in electric potential energy when two protons initially 0.100nm apart are completely separated.

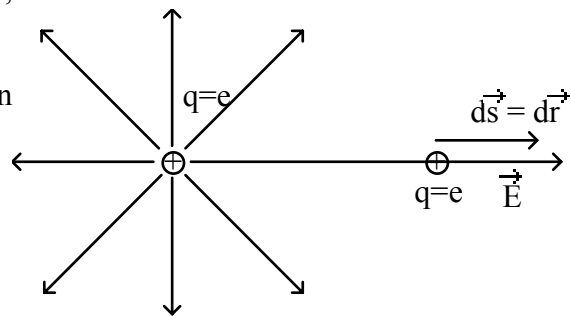
The electric potential energy is,

$$\Delta U = -q \int \vec{E} \cdot d\vec{s}$$

The field is due a point charge and the motion is along the same direction, \hat{r} .

$$\Delta U = -e \int_R^\infty k \frac{e}{r^2} dr = -ke^2 \left[-\frac{1}{r} \right]_R^\infty = -k \frac{e^2}{R}$$

$$\Delta U = -9 \times 10^9 \frac{(1.6 \times 10^{-19})^2}{0.100 \times 10^{-9}} = \underline{\underline{-2.3 \times 10^{-18} \text{ J}}}$$



Note that $\Delta U = U(\infty) - U(R)$. It is customary to set $U(\infty) \equiv 0$ so we can say that the original potential energy of the protons was $U(R) = \underline{\underline{+2.3 \times 10^{-18} \text{ J}}}$. These protons originally had a lot of potential energy when close together and are quite happy to fly apart when given the chance.

Definition of electric potential

A very useful concept in electricity: The electric potential, V , or “voltage” and The electric potential difference between points A and B, $V_A - V_B$.

General Physics II

The electric potential is defined as the potential energy per unit of charge.

$$V = \frac{U}{q}, \text{ where the units are } \frac{\text{Joules}}{\text{Coulomb}} = \text{volts.}$$

Electric potential or voltage is very much like potential energy.

The electric potential difference between points A and B, $V_A - V_B$, is

$$\Delta V = V_A - V_B = \frac{U_A - U_B}{q} = \frac{W_{A \rightarrow B}}{q} = \frac{1}{q} \int_A^B q \vec{E} \cdot d\vec{l},$$

or,

$$\Delta V = V_A - V_B = \int_A^B \vec{E} \cdot d\vec{l}.$$

Then the work and potential can be related as:

$$W_{A \rightarrow B} = U_A - U_B = q\Delta V = q(V_A - V_B).$$

This is the work done by the electric field moving a charge q from A to B.

Example

Find the kinetic energy that an electron released from the negative plate would have just as it reaches the positive plate.

The change in the potential energy of the electron can be found from the definition of potential,

$$\Delta U \equiv q \Delta V = (-1.6 \times 10^{-19} \text{ C})(+100 \text{ V}) = -1.6 \times 10^{-17} \text{ J}$$

Applying the Law of Conservation of Energy,

$$\Delta U + \Delta K = 0 \Rightarrow \Delta U = -\Delta K = -K_f + K_i = -K_f$$

$$\Rightarrow K_f = -\Delta U = +1.6 \times 10^{-17} \text{ J}$$

There is a more convenient unit for energy in problems like this.

Define 1 electron-volt = 1 eV $\equiv 1.6 \times 10^{-19} \text{ J} \Rightarrow$

$$\Delta U = (-1e)(+100 \text{ V}) = -100 \text{ eV and } K_f = -\Delta U = +100 \text{ eV}$$

Potential due to point charges

As we know $\vec{E} = k \frac{q}{r^2} \hat{r}$ and $V(P) = -kq \int_{\infty}^P \frac{\hat{r} \cdot d\vec{r}}{r^2} = -kq \int_{\infty}^P \frac{dr}{r^2} = \frac{kq}{r}$, if we define $V(\infty)=0$.

Then the electric potential due to a point charge

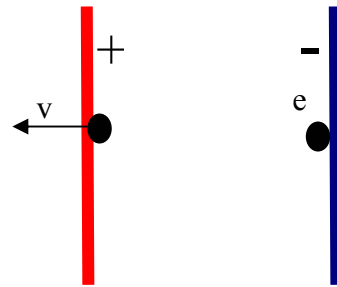
$$V = \frac{kq}{r}$$

It is easy to see that the electric potential due to N charged particles

$$V = \sum_{i=1}^n \frac{kq_i}{r_{0i}}$$

When the charge distribution is continuous the potential can be obtained from the relation:

$$V = k \int \frac{dq}{r}$$



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The electric field and electric potential are very similar except $E \propto \frac{1}{r^2}$ but $V \propto \frac{1}{r}$

The electric field is a vector but the electric potential is a scalar. Therefore, for potential calculation, we don't have to worry about canceling or adding components due to direction (scalar has no direction!)

Examples

Find the potential due to a dipole along its axis at any point in the space.

Since potential is a scalar quantity we can just add up the potentials due to the individual point charges.

$$V = V_+ + V_- = \frac{Q}{4\pi\epsilon_0} \left(\frac{1}{r_1} - \frac{1}{r_2} \right)$$

In practice a is a very small quantity when it compared to r . Distance a is in atomic scale. Therefore it is necessary to make an approximation ($r \gg a$).

The distances can be written as

$$r_1 = \sqrt{r^2 + \left(\frac{a}{2}\right)^2 - ar \cos \theta}; \quad r_2 = \sqrt{r^2 + \left(\frac{a}{2}\right)^2 + ar \cos \theta}$$

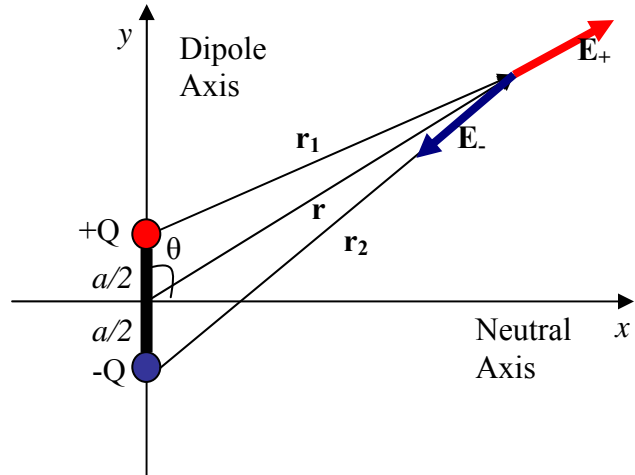
using the identity $(1+s)^n \approx 1+ns$ when $s \ll 1$ we obtain:

$$\frac{1}{r_1} = \frac{1}{r} \left(1 + \frac{1}{2} \frac{a}{r} \cos \theta \right); \quad \frac{1}{r_2} = \frac{1}{r} \left(1 - \frac{1}{2} \frac{a}{r} \cos \theta \right)$$

Then the potential takes the form

$$V = \frac{Qa \cos \theta}{4\pi\epsilon_0 r^2} = \frac{\vec{p} \cdot \hat{y}}{4\pi\epsilon_0 r^2}$$

where \mathbf{p} is the dipole moment. It is defined in chapter 2. It is obvious that on the x-axis the potential is zero.



Example

Find the energy required to assemble all the charges in dipole assuming that there is another charge q on the y-axis.

From the definition of potential, $\Delta U \equiv q \Delta V$, we can find the potential energy needed to bring in each charge to create this distribution. Choose the $+Q$ to be the first one. It can be moved into position using no energy because there are no other charges around yet ($\Delta V=0$). Choose the $-Q$ next,

$$\Delta U_{dipole} = -QV_+ = -\frac{1}{4\pi\epsilon_0} \frac{Q^2}{a} \text{ where } V_+ \text{ is the potential difference felt by the } -Q.$$

Next bring in the charge q .

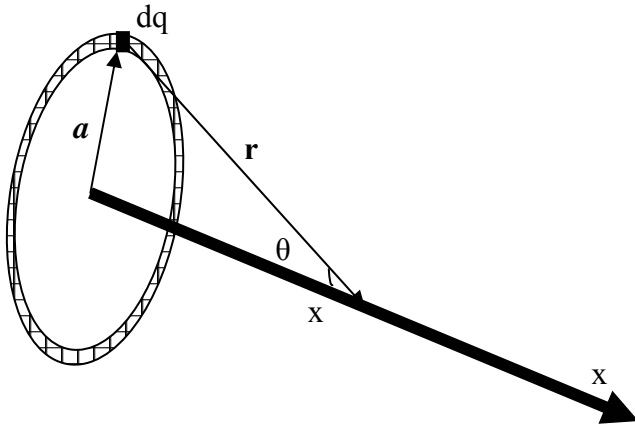
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$$\Delta U_q = q(V_+ + V_-) = qV_{dipole} = \frac{qQa}{4\pi\epsilon_0 y^2}$$

The total energy required is, $\Delta U = \Delta U_q + \Delta U_{dip} = \frac{1}{4\pi\epsilon_0} \left(\frac{qQa}{y^2} - \frac{Q^2}{a} \right)$.

Example

Find the potential due to a ring of charge, Q , and radius, a , a distance, x , from the center along the axis.



Choose a small portion of the ring as a point charge, dq , and use the potential due to a point charge, $dV = k \frac{dq}{r}$. Since r is constant for all dq 's on the ring, the integral is straightforward,

$$\int dV = k \int \frac{dq}{r} \Rightarrow V = k \int \frac{dq}{r} = \frac{k}{r} \int dq = \frac{k}{r} Q$$

In terms of a and x , $V = \frac{1}{4\pi\epsilon_0} \frac{Q}{\sqrt{x^2 + a^2}}$.

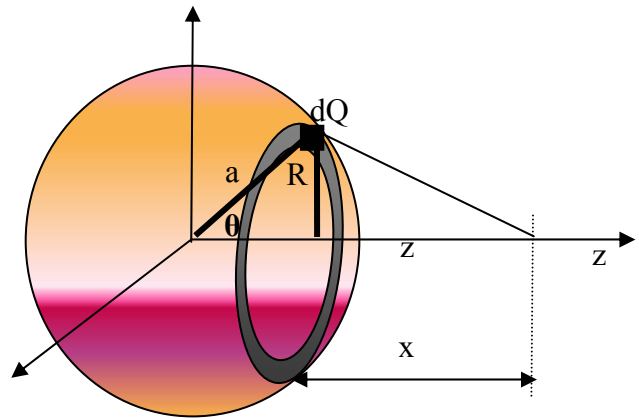
Example

Find the potential due to a spherical shell of radius, a , and charge, Q , a distance, z , from the center of the shell.

Method 1 (requires some mathematical supplement)

Use the rings from example 6 as the dq 's and sum their potentials to find the potential due to the entire sphere. By trigonometry,

$x = z - a \cos \theta$, so the potential, dV , due to the ring at an angle θ is, $dV = k \frac{dQ}{(R^2 + [z - a \cos \theta]^2)^{1/2}}$.



Note that $R = a \sin \theta$, so,

$$dV = k \frac{dQ}{(a^2 \sin^2 \theta + z^2 - 2za \cos \theta + a^2 \cos^2 \theta)^{1/2}} \quad dV = k \frac{dQ}{(a^2 + z^2 - 2za \cos \theta)^{1/2}}$$

Since the charge is distributed uniformly over the surface,

$$\frac{dQ}{Q} = \frac{\text{area of the strip}}{\text{area of the sphere}} = \frac{2\pi R a d\theta}{4\pi a^2} = \frac{2\pi a^2 \sin \theta d\theta}{4\pi a^2} \Rightarrow dQ = \frac{Q}{2} \sin \theta d\theta.$$

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Now dV can be written entirely in terms of θ , $dV = k \frac{Q \sin \theta \, d\theta}{2(a^2 + z^2 - 2za \cos \theta)^{3/2}}$.

The integration can be completed by letting $u = a^2 + z^2 - 2za \cos \theta \Rightarrow du = 2za \sin \theta \, d\theta$.

So, $\int_0^V dV = \frac{kQ}{4az} \int_{\theta=0}^{\pi} u^{-1/2} du \Rightarrow V = \frac{kQ}{4az} \cdot 2 \left[(a^2 + z^2 - 2za \cos \theta)^{1/2} \right]_0^{\pi}$

Finally, $V = \frac{kQ}{2az} \left[\sqrt{(a+z)^2} - \sqrt{(a-z)^2} \right]$.

There are two cases: $V = \begin{cases} \text{if } z > a & V = \frac{kQ}{2az} [(a+z) - (z-a)] = k \frac{Q}{z} \\ \text{if } z < a & V = \frac{kQ}{2az} [(a+z) - (a-z)] = k \frac{Q}{a} \end{cases}$

Notice that the potential is constant inside the sphere which is consistent with the constant field we expect. It falls off as the potential of a point charge outside.

Method 2: In this system the electric field can easily be calculated from the Gauss law. (see chapter 3).

$$\vec{E} = 0; \quad r < R \quad E = \frac{Q}{4\pi\epsilon_0 r^2}; \quad r > R$$

Therefore we can determine the potential by using the equation:

$$\Delta V = V_A - V_B = \int_A^B \vec{E} \cdot d\vec{l}$$

Since the potential is zero at infinity $V_A(\text{infinity})$, in order to calculate potential outside the sphere:

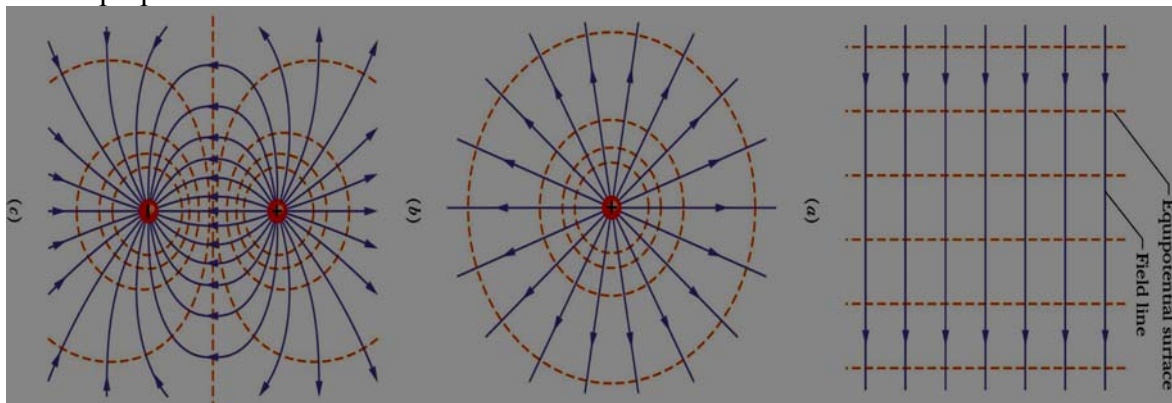
$$V_{out} = - \int_{\infty}^r \vec{E}_{out} \cdot d\vec{l} = - \int_{\infty}^r \frac{Q}{4\pi\epsilon_0 r^2} dr = \frac{Q}{4\pi\epsilon_0 r}$$

Similarly the potential inside the sphere is

$$V_{in} = - \int_{\infty}^a \vec{E}_{out} \cdot d\vec{l} - \int_a^{r_m} \vec{E}_{in} \cdot d\vec{l} = - \int_{\infty}^a \frac{Q}{4\pi\epsilon_0 r^2} dr = \frac{Q}{4\pi\epsilon_0 a}$$

Equipotential Surfaces

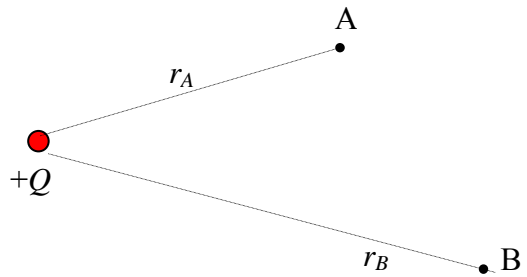
For a point charge, find the equipotential surfaces. First draw the field lines, then find a surface perpendicular to these lines.



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Exercises

1. Find the electric potential difference (or the voltage) between two points, A and B, in the electric field produced by a point charge $+Q$.



a. What position is at a higher potential, or higher voltage?

b. Usually a position is chosen where the potential is defined to be zero ($V=0$). (This is very much like choosing the position where the potential energy is zero.) In the case of a spherical charge distribution, this position is taken as infinity (∞). If this is the case, then what is the *electric potential* at a distance r from the charge Q ?

c. What is the potential difference if the point charge is negative ($-Q$)?

d. What position is at a higher potential, or higher voltage?

2. Find the value of the electric potential at distances 1 m, 2 m, and 3 m in various directions from the point charge $+Q$. Take $Q = +2 \mu\text{C}$. Note that $V_A - V_B = \int \vec{E} \cdot d\vec{l}$ means that in

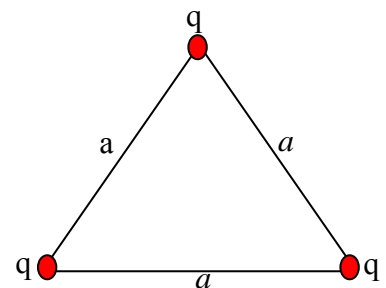
order for the potential to be the same (no potential difference) at two points on a path, the path must be perpendicular to the E-field. The path on which this occurs is called an equipotential. Draw the electric field lines of force and note their relationship to the equipotentials.

3. What happens to the values of the electric field and equipotentials if Q is negative?

4. Two charges, $q_1 = +2 \mu\text{C}$ and $q_2 = -5 \mu\text{C}$, are placed 4 cm apart on the x -axis. Find the position(s) on the x -axis where the electric potential is zero.



5. How much work must *you* do in order to move three charges from infinity to the corners of an equilateral triangle with sides of length a ?



6. For two large, oppositely charged parallel plates separated by a distance d , what is the potential difference between the plates? What should the equipotentials look like? How about the electric field lines of force?

8. Find the electric potential both outside and inside a solid sphere of radius R with a charge density $\rho_v = \text{constant}$. Choose $V = 0$ at infinity.

9. A charge, Q , is uniformly distributed on a rod of length L . Find the electric potential a distance y on the perpendicular bisector of the rod.

10. Find the potential on a line passing through the center of a disk perpendicular to the disk where the disk has a uniform charge density ρ_s .

Electric field calculations from the electric potential

Remember that the potential difference between two points is related to the electric field as:

$$V_A - V_B = \int_A^B \vec{E} \cdot d\vec{l}$$

When rearranged:

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$$V_B - V_A = - \int_A^B \vec{E} \cdot d\vec{l} = - \int_A^B E_s ds.$$

Take the differential of this expression and rearrange.

$$E_s = - \frac{dV}{ds}.$$

Example

Find the axial electric field (electric field on the dipole axis) due to a dipole.

The potential of the dipole on the axis can be obtained by setting $\theta=0$. $V = \frac{p}{4\pi\epsilon_0 r^2}$ The

electric field along the x-axis can be found by applying $E = - \frac{dV}{dr} = \frac{p}{2\pi\epsilon_0 r^3}$ as we got before.

Example

Find the field on the axis of the ring of charge Q.

From example 6, $V = k \frac{Q}{(R^2 + x^2)^{3/2}}$. The electric field along the x-axis can be found by

applying $E_s = - \frac{dV}{ds} \Rightarrow E_x = - \frac{dV}{dx} = -kQ \left\{ -\frac{1}{2} (R^2 + x^2)^{-3/2} (2x) \right\} = k \frac{Qx}{(R^2 + x^2)^{3/2}}$ as we

obtained before!

Charge sharing

Suppose *conductors* are brought into contact or connected by wires. In the electrostatic situation, the E field in the conductors will be zero, and therefore, all points in and on the conductors will be at the same potential. This idea allows us to find how the charge density, ρ_s , varies as a function of the curvature of the conducting surface.

Exercise: Two solid conducting spheres of radii $R_1 = 10$ cm and $R_2 = 20$ cm have charges of $+2 \mu\text{C}$ and $-5 \mu\text{C}$, respectively. The spheres are brought into contact, then separated again. How much charge is on each sphere, what is the surface charge density on each sphere, and what is the potential of each sphere?

Electrostatic Generator

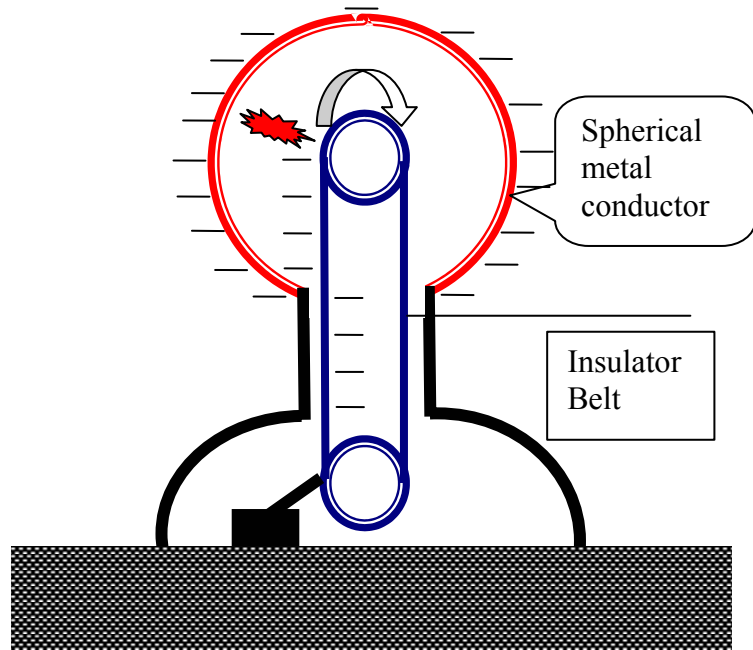
An **electrostatic generator** is a mechanical device can produce continuous current. The knowledge of static electricity dates back to the earliest civilizations, but for millennia it remained merely an interesting and mystifying phenomenon. The development of electrostatic machines did not begin in earnest until the 18th Century. Electrostatic generators operate by using manual (or other) power to transform mechanical work into electric energy. Electrostatic generators develop electrostatic charges of opposite sign rendered to divided conductors. These devices can produce high voltage electrical output at relatively low electrical currents. Electrostatic generators are of two kinds: (A) **Frictional machines**, and (B) **Influence machines**.

Example

General Physics II

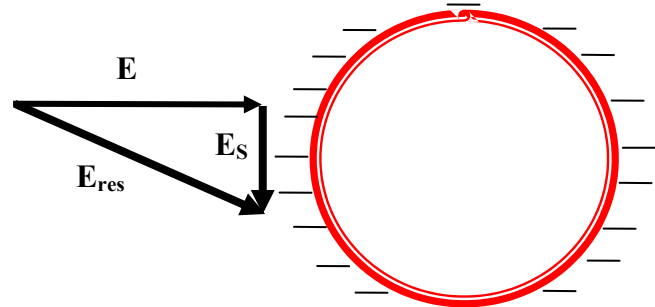
Lightning production.

In the high voltage electrostatic generator electric charges are carried by the belt, made of a special insulating material, and charge a large metallic sphere with radius $R=1.5$ m, as it is shown in Fig. Width of the belt is $l=1$ m, its speed is 20 m/s. A discharge in the air takes place when the strength of electric field exceeds $E=30$ kV/cm. Estimate the maximal values of voltage and current that can be obtained with such generator.



Solution

The shown generator is named a Van de Graaff generator. This is a huge device that occupies a room with height of several storeys. It is used to obtain a very significant voltage. A maximal voltage between the Earth and a charged sphere in the generator depends upon the air conductivity. If electric field by the sphere surface exceeds the given value of $E = 30$ kV/cm, charge leaks off in sparks between the sphere and some nearest grounded points. The same phenomenon in the atmosphere we observe between a cloud and the Earth during a storm. The field around the sphere is neither uniform, nor symmetric because a sphere is not a sole conductor in the space. Thus it is difficult to calculate accurately a voltage or potential distribution around the sphere. We can only estimate voltage between the sphere and the Earth. First supposition must determine how to consider the electric field of the Earth. We know that our planet is charged negatively, is surrounded by the electric field with $E_s \approx -130$ V/m = -1.3 V/cm on its surface, and has non-zero potential. It can be easily shown that we should not take into account the electric field of the Earth. Electric field is a vector field. Let us draw a vector of the resultant field just by the sphere surface. Suppose we have chosen a point very close to the sphere. The resultant vector of electric field E_{res} at this point has almost the same magnitude and direction as in the absence of the Earth due to the value of $E / E_{res} = 23000$. For any point by the sphere surface we can come to the same conclusion.



In particular, it means that for our estimation the field around the sphere can be calculated as a field of a solitary sphere in the space. If we had only a sole charged sphere in the empty space, its field E_1 and potential φ_1 would be related as for the field of a point charge, i.e.:

$$E_1 = \frac{\varphi_1}{R} \quad (1)$$

According to eq.(1) the maximal potential of the charged sphere in generator is equal to:

$$\varphi_{max} = E.R \approx -4.6 \times 10^5 \text{ V} \quad (2)$$

Eq.(2) estimates also the maximal voltage as we neglected other charged objects in surroundings.

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The maximal current I_{\max} is the maximal rate of charge deposition from the belt on the sphere. It is also determined by the air conductivity. Electric breakdown between the charged belt and the Earth can occur before the sphere is charged critically.

To estimate the maximal current we can assume the belt to be an infinite plane with its surface charge density ρ_s . From the relationship between E , ϵ_0 (electric permittivity), and ρ_s for a plane the maximal value of charge density $\rho_{s,\max}$ is following:

$$\rho_{s,\max} = 2\epsilon_0 E \quad (3)$$

Using eq.(3) it is easy to calculate the maximal current:

$$I_{\max} = \rho_{s,\max} v l = 2\epsilon_0 E v l = 10^{-3} \text{ Amperes} \quad (4)$$

The calculated results clarify why the generator is usually very big.

kelvin's water drop electrostatic generator



Description:

Control the flow of water such that the water drops break just above the corona. To begin, there must exist a charge imbalance. As the water drops fall through the metal loops, electrostatic charge builds. One of the loops is positive and the other is negative. Once enough charge has been collected, current begins to flow, lighting one or more of the lights along the middle front edge of the apparatus.