# Chapter 3: Gauss' Law

We now want to quickly discuss one of the more useful tools for calculating the electric field, namely Gauss' law. In order to understand Gauss's law, it seems we need to know the concept of electric flux. We can image the flux as a volume rate of flow. We also can imagine the light come out to the room is the same as the light come out from the light bulk.

Electric Flux

Flux comes from the Latin word meaning "flow". Field lines roughly describe the electric field strength. The strength of electric field is represented by the relative number of field lines passing through a unit area:

$|\vec{E}| \propto$ number of field lines/area
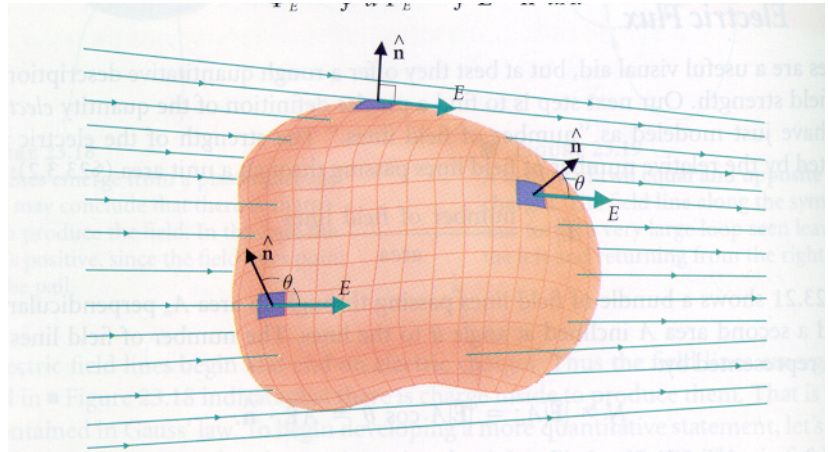
$\phi_i = \vec{E} \cdot \Delta\vec{A}_i = EA_i \cos\theta$

for whole surface:

$\phi = \sum \vec{E} \cdot \Delta\vec{A}_i$

$\Delta\vec{A}_i \rightarrow$ small $\rightarrow d\vec{A}$ for continuous surface

Now we have $\phi = \oint \vec{E} \cdot d\vec{A}$ where $\oint$ is a whole surface integral.
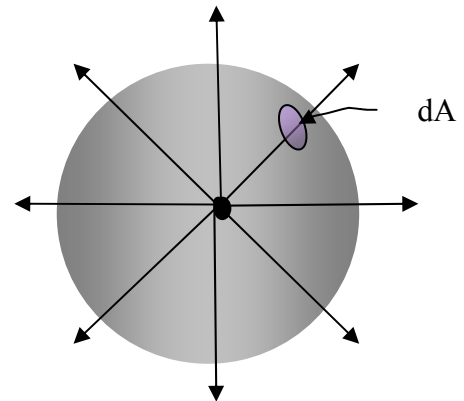
Example

Find the flux that exits a sphere centered at the origin due to a point charge also at the origin. A small surface area on a sphere is,

$d\vec{A} = r^2 \sin\theta \, d\theta \, d\phi \, \hat{r}$. Using the field due to the point charge, $\vec{E} = k\dfrac{q}{r^2}\hat{r}$, the flux can be calculated from its definition, $\Phi \equiv \int \vec{E} \bullet d\vec{A}$,

$\Phi \equiv \int k\dfrac{q}{r^2}\hat{r} \bullet r^2 \sin\theta \, d\theta \, d\phi \, \hat{r}$

$\Phi = k \, q \int_{0}^{\pi} \sin\theta \, d\theta \int_{0}^{2\pi} d\phi = 4\pi k q = \underline{\underline{\dfrac{q}{\varepsilon_o}}}$
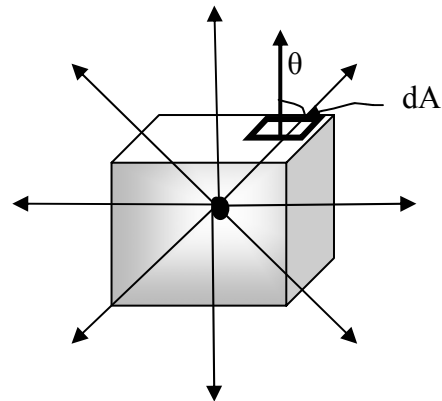
Example

Find the flux due to a point charge over the surface of a cube. Again start with the definition of flux, $\Phi \equiv \int \vec{E} \bullet d\vec{A}$, and the field of a point charge, $\vec{E} = k\dfrac{q}{r^2}\hat{r}$. This time however, the electric field and the area element are not always parallel and the integral is very difficult. But, we know that the number of field lines that leave the charge is *not affected by the shape of some imaginary surface that surrounds the charge*.

General Physics II

That is, the same number of field lines exit the cube as exit the sphere in previous example so the flux must again equal $\Phi = \dfrac{q}{\varepsilon_o}$.

(Optional) In order to calculate flux through the cube due to point charge we can write the electric field in Cartesian coordinate system:

$$\vec{E} = \frac{q}{4\pi\varepsilon_0} \frac{x\hat{x} + y\hat{y} + z\hat{z}}{\left(x^2 + y^2 + z^2\right)^{3/2}}$$

Let us calculate the flux through the upper surface. The surface area is $d\vec{A} = dxdy\hat{z}$, the the flux is

$$\Phi_{upper} = \frac{q}{4\pi\varepsilon_0} \iint \frac{z\,dxdy}{\left(x^2 + y^2 + z^2\right)^{3/2}}$$

I have evaluated this integral by using Mathematica(A computer program) and the result is $\Phi_{upper} = \dfrac{q}{6\varepsilon_0}$. We have six surface the the total flux is $\Phi = \dfrac{q}{\varepsilon_0}$.

Gauss'Law

A formal statement of the relation between charge and flux is known as Gauss' law.

> The total electric flux emerging from an arbitrary volume equals the net charge enclosed within the volume divided by ε_0.
>
> $$\phi \equiv \oint \vec{E} \cdot d\vec{A} = \frac{Q_{enclosed}}{\varepsilon_0}$$
>
> **Gauss' Law**

Gauss' law is a powerful tool to calculate electric field. However, it's only useful when $\vec{E}$ can be taken outside of integral. That means electric field $\vec{E}$ is constant and symmetrical for the chosen surface (e.g., the line is very long or the plane is very big, so that it's no effect from the edges) . For example, if $\vec{E}$ parallel $d\vec{A}$ and constant then

$$\oint \vec{E} \cdot d\vec{A} = E \oint d\vec{A} = \frac{Q_{enclosed}}{\varepsilon_0}$$

so now we can find electric field $\vec{E}$ easily by

$$E = \frac{Q_{enclosed}}{\varepsilon_0 \oint d\vec{A}}$$

So the chosen surface is very important to make $\vec{E} \parallel d\vec{A}$ so $\vec{E}$ can be pulled out of the integral!

For continuous charge distributions the enclosed charge can be written as:

$Q_{enc} = \int \rho_\ell d\ell$ for line, $Q_{enc} = \int \rho_s dS$ for surface, $Q_{enc} = \int \rho_v dv$ for vulume   charge distributions.

In summary, Gauss's Law is usually used in either of two ways:
1) Given the field and the surface then enclosed charge can be found.
2) Given the enclosed charge and sufficient symmetry to choose a convenient surface, then the field can be found.

General Physics II

A note:

Conductor: A material in which electrons are completely free to move in response to applied electric fields. In a conductor the electrons will move until they find a place where the field is zero. Therefore, the field inside a conductor will always be zero, if you wait long enough for the electrons to find these places. Typically, this takes microseconds.

Examples

Spherical symmetry

    a) Given a conductor sphere of radius R with a total charge, Q, find the electric field inside and outside the sphere.

Gauss surface (imaginary)

R

r

Charge Q distributed over the surface of the conductor!
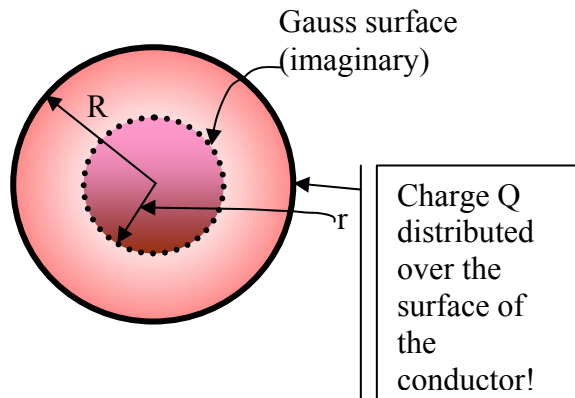
Imagine a surface (not a real(physical) surface it is a mathematical surface) just inside the real surface of the conductor over which we will apply Gauss' Law. Since this "Gaussian surface" is inside a conductor the charge inside the conductor is zero, therefore

$$\Rightarrow \oint E \bullet dA = 0 \Rightarrow \frac{q_{encl}}{\varepsilon_o} = 0 \Rightarrow q_{encl} = 0 . \ \vec{E} = 0; \quad r < R$$

The gaussian surface can be made arbitrarily close to the real surface.

Gauss surface (imaginary)

R

Normal of the surface paralel to the electric field

r   **E**
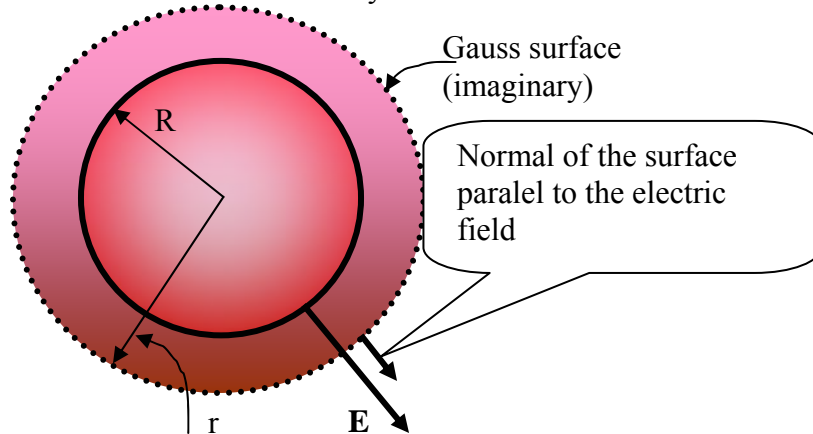
Electric field outside the sphere can be calculated:

$$\int_S \vec{E}.d\vec{S} = \frac{Q}{\varepsilon_0} \Rightarrow E4\pi r^2 = \frac{Q}{\varepsilon_0} \Rightarrow E = \frac{Q}{4\pi\varepsilon_0 r^2}; \quad r > R .$$

    b) A total charge, Q, is uniformly distributed throughout a non-conducting sphere of radius, R. Find the electric field inside and out. Sketch E vs. r.

    The spherical symmetry means that E will be constant over any concentric gaussian sphere and E will point radially (parallel with dA). Therefore the flux integral

Gauss surface (imaginary)

R

r   **E**
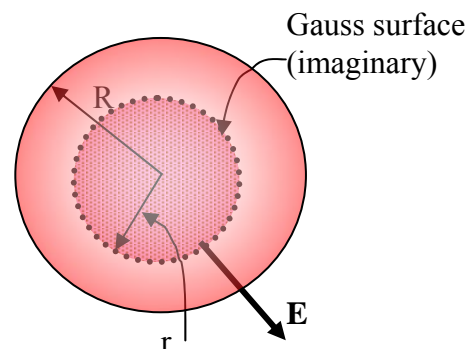
in Gauss' Law is, $\oint \vec{E} \bullet d\vec{A} = E\,A = E(4\pi r^2)$. For a gaussian sphere with r<R the charge enclosed is proportional to the fraction of the volume of the real sphere that the gaussian sphere occupies.

$$q_{encl} = \frac{\frac{4}{3}\pi r^3}{\frac{4}{3}\pi R^3}Q = \frac{r^3}{R^3}Q. \text{ Applying Gauss' Law, } \oint \vec{E}\bullet d\vec{A} = \frac{q_{enclosed}}{\varepsilon_o},$$
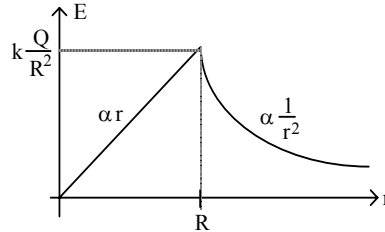
$$E(4\pi r^2) = \frac{r^3}{R^3}\frac{Q}{\varepsilon_o} \Rightarrow E = k\frac{Qr}{R^3} \quad r<R$$

For a gaussian sphere with r>R the charge enclosed is just the total charge, Q. Applying Gauss' Law, $\oint \vec{E} \bullet d\vec{A} = \frac{q_{enclosed}}{\varepsilon_o} \Rightarrow E(4\pi r^2) = \frac{Q}{\varepsilon_o} \Rightarrow E = k\frac{Q}{r^2}$ r > R This
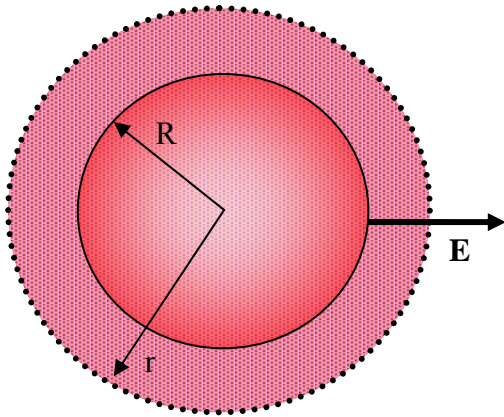
field is just the field due to a point charge Q at the origin. Notice the equations for the fields inside and outside agree at r=R as they must.

Try the problems including electric fields of concentric spheres with various charge distributions. i.e is charge distribution of a sphere is $\rho_v = \rho_0(1+r)$ can you calculate electric field inside and outside the sphere??

Cyclindrical symmetry

a) A very long thin wire has a linear charge density, ρ_ℓ, uniformly distributed throughout its length. Find the electric field as a function of the distance.

By symmetry the electric field must point radially outward and it can only depend on r. Therefore, the best gaussian surface is a concentric cylinder of radius, r. Using Gauss Law

$$\int_S \vec{E}.d\vec{S} = \frac{q_{enc}}{\varepsilon_0} \Rightarrow E2\pi rL = \frac{1}{\varepsilon_0}\int_0^L \rho_\ell d\ell$$

$$E = \frac{\rho_\ell}{2\pi\varepsilon_0 r}$$

b) A very long non-conducting cylinder of radius, a, has a charge density, ρ_ℓ, uniformly distributed throughout its volume. Find the electric field as a function of the distance from the axis, r, and sketch a graph of E vs. r.
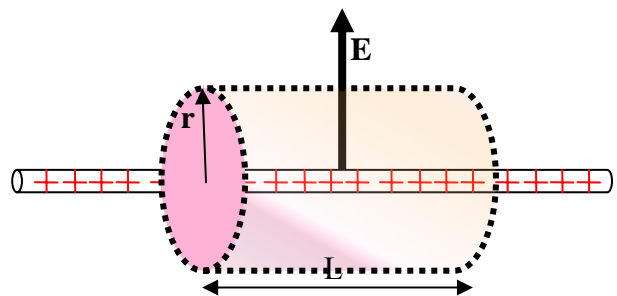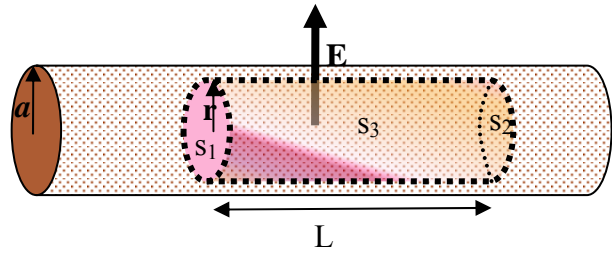
General Physics II

By symmetry the electric field must point radially outward and it can only depend on r. Therefore, the best gaussian surface is again a concentric cylinder of radius, r.

The flux integral in Gauss' Law can be broken up into three parts. One for each surface shown,

$$\oint \vec{E} \bullet d\vec{A} = \int_{s_1} \vec{E} \bullet d\vec{A} + \int_{s_2} \vec{E} \bullet d\vec{A} + \int_{s_3} \vec{E} \bullet d\vec{A}$$

The integrals over s_1 and s_2 are both zero because the field is radial which means that no flux exits these faces (E is perpendicular to dA). Over s_3 E is constant and parallel to dA so that integral is, $\int_{s_3} \vec{E}.d\vec{S} = \frac{q_{enc}}{\varepsilon_0} \Rightarrow E2\pi r L = \frac{q_{enc}}{\varepsilon_0}$. Be careful when you calculate the enclosed charge. The total charge in a length L of the cylinder is: $q_L = \rho_\ell \pi a^2 L$. Then for r<a the charge enclosed in the gaussian surface is, $q_{enc} = \rho_\ell L \frac{\pi r^2 L}{\pi a^2 L} = \rho_\ell L \frac{r^2}{a^2}$. The electric field will be $E = \frac{\rho_\ell}{2\pi\varepsilon_0} \frac{r}{a^2}; \quad r < a$

For r>a the charge enclosed in the gaussian surface is, $q_{enc} = \rho_\ell L$. Applying Gauss' Law,

$$\int_{s_3} \vec{E}.d\vec{S} = \frac{q_{enc}}{\varepsilon_0} \Rightarrow E2\pi r L = \frac{\rho_\ell L}{\varepsilon_0} \Rightarrow E = \frac{\rho_\ell}{2\pi\varepsilon_0 r};$$

Note that the field falls off as $\frac{1}{r}$ not $\frac{1}{r^2}$ as you might expect. The equations for the fields inside and outside agree at r=a as they must.

Try yourself calculation of the electric field inside and out side the concentric cylinders! Also try to the calculation of electric field of conductor cylinder!
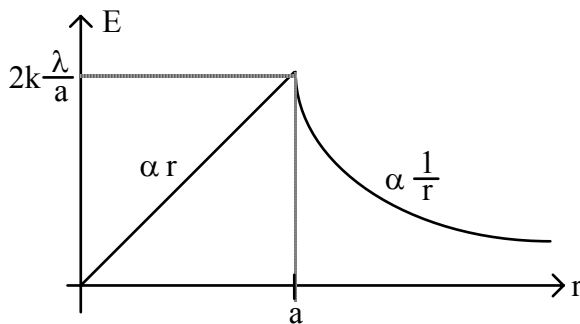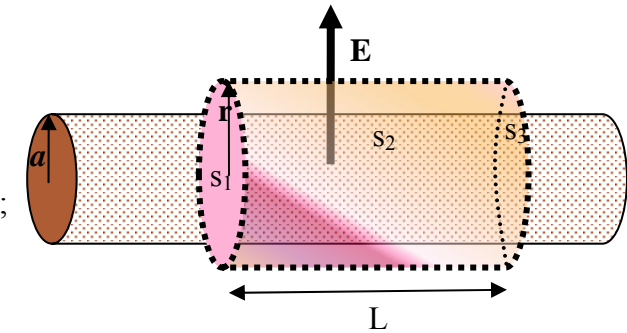
Cartesian symmetry

A large metal plate of thickness, 2t, has a uniform charge density, σ. Find the electric field as a function of the distance from the center of the plate, z.

By symmetry the E-field can only be a function of z and it must point directly away from the plate. Inside the plate the field is zero because it is a conductor. Outside we can choose the gaussian surface shown.

The surfaces s_1 and s_2 are the same arbitrary shape, parallel with the surface of the plate, and equal distances from it. The gaussian surface is completed with the surface s_3 which is everywhere perpendicular to the surface of the plate. The flux leaving the gaussian surface is,

$$\oint \vec{E} \bullet d\vec{A} = \int_{s_1} \vec{E} \bullet d\vec{A} + \int_{s_2} \vec{E} \bullet d\vec{A} + \int_{s_3} \vec{E} \bullet d\vec{A} .$$

Since the field is only along the z-axis the integral over s_3 is zero. The magnitude of the field can only depend on z, so the fields at s_1 and s_2 are equal and constant so the integrals are straightforward,

$$\oint \vec{E} \bullet d\vec{A} = EA + EA + 0 = 2EA$$

The charge enclosed in the gaussian surface is, $q_{encl} = \sigma A$. Applying Gauss' Law,

$$\oint \vec{E}.dA = \frac{q_{enc}}{\varepsilon_0} \Rightarrow 2EA = \frac{\sigma A}{\varepsilon_0} \Rightarrow E = \frac{\sigma}{2\varepsilon_0}$$

Try electric fields of the parallel plates with different charge distributions.

<u>Remember again</u>

The Definition of Electric Flux $\Phi \equiv \int \vec{E} \bullet d\vec{A}$

Gauss' Law $\oint \vec{E} \bullet d\vec{A} = \dfrac{q_{enclosed}}{\varepsilon_o}$

The Behavior of Conductors,

       1) E=0 everywhere inside.

       2) Excess charge stays on the surface.

       3) $E = \dfrac{\sigma}{\varepsilon_o}$ and perpendicular to the surface just outside.