11. INDUCTANCE

11-1 Self Induction

If two coils near each other, a current i in one coil will set up a flux Φ_B through the second coil. If this flux changes in time an induced emf will appear in the second coil according to Faraday's Law. This is called *self induction* and the emf produced is called *self induced emf* (or *back emf*). For the second coil:

$$\varepsilon = -\frac{d(N\Phi_B)}{dt}$$

here the product $N\Phi_B$ is the number of flux linkages (N being number of turn).

For a given coil (if no magnetic materials such as iron nearby) this quantity is proportional to to the current i.

$$N\Phi_B = Li$$

or

$$\boxed{L = \frac{N\Phi_B}{i}}$$

where L is a proportionality constant – called the *inductance* of the coil. This is analogous to the defining the capacitance:

$$C = \frac{Q}{V}$$

The inductance depends on the geometry of the device. Hence:

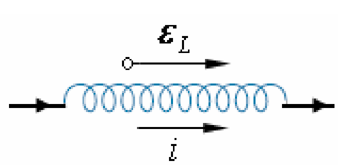
$$\varepsilon = -\frac{d(N\Phi_B)}{dt} = -L\frac{di}{dt}$$

or

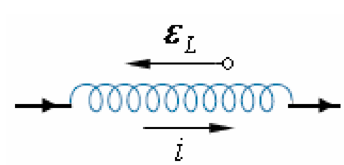
$$\boxed{L = -\frac{\varepsilon}{di/dt}}$$

The SI unit of L is Henry (H) → 1 H = 1 V·s/A

We can find the direction of a self-induced emf from Lenz' Law.
Assume that he instantaneous currents below are the same in ach case self induced emf ε_L is directed so as to oppose the change.



the current is decreasing



the current is increasing

EXAMPLE 1

Find the inductance of a uniformly wound solenoid having N turns and length ℓ. Assume that ℓ is much longer than the radius of the windings and that the core of the solenoid is air.

SOLUTION

We can assume that the magnetic field inside a solenoid is uniform. Thus:

$$B = \mu_0 n i = \mu_0 \frac{N}{\ell} i$$

where $n = N/\ell$ is the number of turns per unit length. The flux is:

$$\Phi_B = BA = \mu_0 \frac{NA}{\ell} i$$

where A is the cross-sectional area of the solenoid. Finally, the inductance is:

$$L = \frac{N\Phi_B}{i} = \frac{\mu_0 N^2 A}{\ell}$$

or if we use $N = n\ell$

$$\boxed{L = \mu_0 n^2 A \ell}$$

Ques: What would happen to the inductance if a ferromagnetic material were placed inside the solenoid?

Ans: The inductance would increase. For a given current, the magnetic flux is now much greater because of the increase in the field originating from the magnetization of the ferromagnetic material. For example, if the material has a magnetic permeability of 500 μ_0 , the inductance would increase by a factor of 500.

EXAMPLE 2

(a) Calculate the inductance of an air-core solenoid containing 300 turns if the length of the solenoid is 25.0 cm and its cross-sectional area is 4 cm^2.

(b) Calculate the self-induced emf in the solenoid if the current through it is decreasing at the rate of 50 A/s.

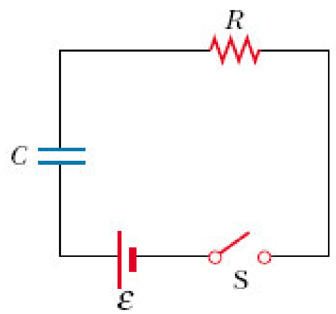
SOLUTION

(a) $L = \dfrac{\mu_0 N^2 A}{\ell} = \dfrac{(4\pi \times 10^{-7})(300)^2(4\times 10^{-4})}{25\times 10^{-2}} = 1.81\times 10^{-4}$ T·m^2/A = 0.181 mH

(b) $di/dt = -50$ A/s → $\varepsilon_L = -L\dfrac{di}{dt} = -(1.81\times 10^{-4})(-50) = 9\times 10^{-3}$ V = 9 mV

11-2 RL Circuits

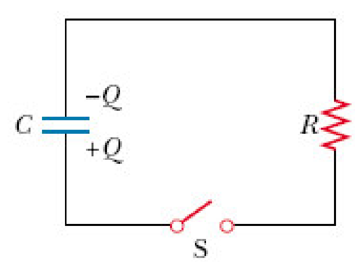
<u>Charging a capacitor</u>



An external emf ε connected to a single RC circuit, charges the capacitor. Its charge in any time *t* is given by:

$$q = C\varepsilon(1 - exp(-t/\tau))$$

<u>Discharging a capacitor</u>

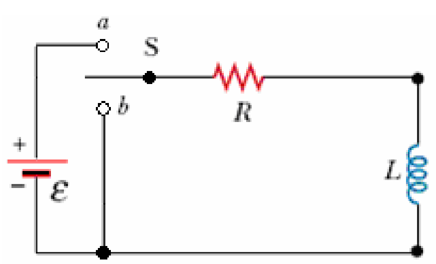


If we remove battery:

$$q = C\varepsilon(exp(-t/\tau))$$

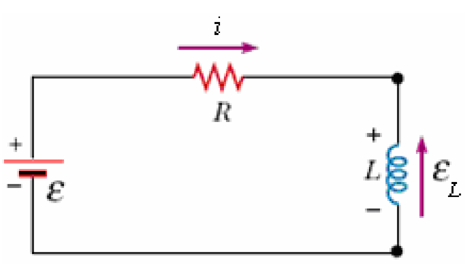
With the time constant of the circuit $\tau = RC$

A basic analogy can be done for an RL circuit.
If a circuit contains a coil, such as a solenoid, the self-inductance of the coil prevents the current in the circuit from increasing or decreasing instantaneously. A circuit element that has a large self-inductance is called an inductor and has the circuit symbol 



Consider an RL circuit given in Figure and the switch *S* closed on position *a* at $t = 0$ s.

Position *a*



$\varepsilon_L < 0$ since $di/dt > 0$

The current in the circuit begins to increase and a back emf (ε_L) that opposes the increasing current is induced in the inductor.

$$\varepsilon_L = -L\frac{di}{dt}$$

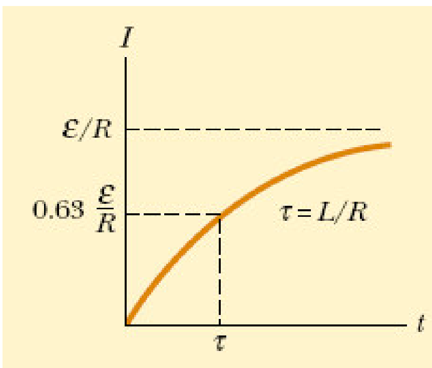
Loop Theorem: $\mathcal{E} - iR - L\frac{di}{dt} = 0 \;\longrightarrow\; \mathcal{E} = iR + L\frac{di}{dt}$

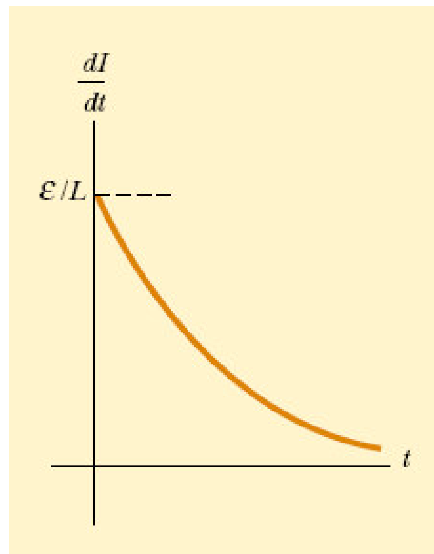
Solution of this DE is: $i = \frac{\mathcal{E}}{R}(1 - e^{-Rt/L})$

We can also write this expression as: $$\boxed{i = \frac{\mathcal{E}}{R}(1 - e^{-t/\tau})}$$

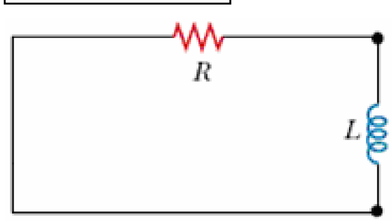
where $\tau = L/R$ is the time constant of the circuit.

Time rate of change of the current is: $\frac{di}{dt} = \frac{\mathcal{E}}{L}e^{-t/\tau}$





Position *b*



If the switch S, having been left in position *a* long enough for the equilibrium current (ε / R) to be established, is thrown to *b*.

Thus, we have a circuit with no battery ($\varepsilon = 0$).

Loop Theorem: $iR + L\frac{di}{dt} = 0$

Solution of this DE is: $$\boxed{i = \frac{\mathcal{E}}{R}e^{-t/\tau}}$$

Time rate of change of the current is: $\frac{di}{dt} = \frac{\mathcal{E}}{L}e^{-t/\tau}$

EXAMPLE 3

A solenoid has an inductance L=50 H and a resistance R = 30 ohm. If it is connected to a 100-V battery, how long will it take for the current to reach its final equilibrium value?

SOLUTION

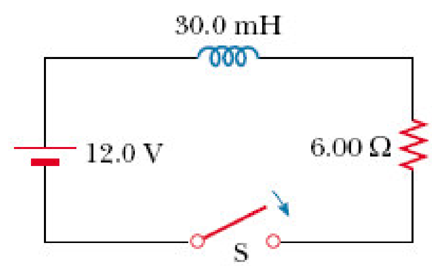
The equilibrium value of the current is reached as $t \to \infty$; that is $i(\infty) = \varepsilon / R$. Hence:

$$\frac{1}{2}\frac{\varepsilon}{R} = \frac{\varepsilon}{R}(1 - e^{-t/\tau}) \quad \rightarrow \quad t = \tau \times \ln 2 = \frac{L}{R} \times 0.693 = \frac{50}{30} \times 0.693 = 1.2\ \mathrm{s}$$

EXAMPLE 4

The switch in Figure is thrown closed at $t = 0$.

(a) Find the time constant of the circuit.
(b) Calculate the current in the circuit at $t = 2$ ms.
(c) Calculate the current in the circuit and the voltage across the resistor after a time interval equal to one time constant has elapsed.



SOLUTION

(a) $$\tau = \frac{L}{R} = \frac{30.0 \times 10^{-3}\ \mathrm{H}}{6.00\ \Omega} = \boxed{5.00\ \mathrm{ms}}$$

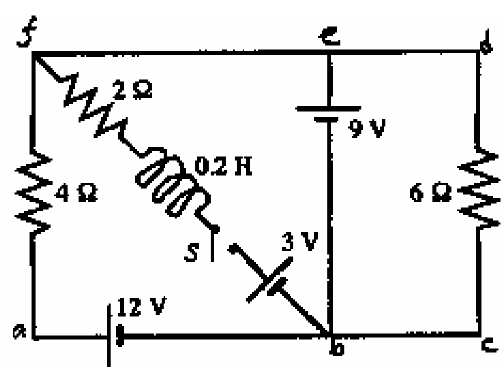
(b) $$I = \frac{\mathcal{E}}{R}(1 - e^{-t/\tau}) = \frac{12.0\ \mathrm{V}}{6.00\ \Omega}(1 - e^{-0.400}) = \boxed{0.659\ \mathrm{A}}$$

(c) $$i = \frac{\varepsilon}{R}(1 - e^{-t/\tau}) = \frac{12}{6}(1 - e^{-1}) = 1.26\ \mathrm{A} \quad \text{and} \quad V = iR = (1.26)(6) = 7.56\ \mathrm{V}$$

EXAMPLE 5

The circuit shows three ideal batteries, three resistors and an ideal inductor. The switch S is closed at t = 0.

(a) Find the potential difference across the inductor, just after the switch is closed.
(b) Determine the current through the inductor, a long time after the switch is closed (that is: $t \to \infty$)



SOLUTION

(a) Initially an ideal inductor acts as an open circuit at t =0 (inductor acts to oppose changes in the current through it).

For loop bfeb:
$+3 + V_L(t=0) - 9 = 0 \rightarrow V_L(t=0) = 6$ V

(b) A long time later, an ideal inductor behaves like a short circuit.
Thus $V_L(t = \infty) = 0$. Assuming the current in the inductor is i:

For loop befb:
$+9 - 2i - 3 = 0 \rightarrow i = 3$ A

11-3 Energy in a Magnetic Field

We have:

$$\varepsilon = iR + L\frac{di}{dt} \quad \text{(loop thorem)}$$

multiply each term by *i*:

$$i\varepsilon = i^2 R + Li\frac{di}{dt} \quad \text{(conservation of energy)}$$

The rate at which energy supplied by the battery ($i\varepsilon$)	=	The rate at which energy delivered to the resistor (i^2R)	+	The rate at which energy stored in the inductor ($Lidi/dt$)

Let U denote the total energy stored in the inductor at any time t.

we can write the rate dU/dt at which energy is stored as: $\frac{dU}{dt} = Li\frac{di}{dt}$

Total energy : $U = \int dU = \int_0^i Lidi = L\int_0^i idi$

solving the integral: $\boxed{U = \frac{1}{2}Li^2}$

*** This expression is similar to the energy stored in a capacitor: $U = \frac{1}{2}\frac{q^2}{C}$

We can also determine the energy density of a magnetic field.
For simplicity, consider a solenoid:

Inductance: $L = \mu_0 n^2 A\ell$

Magnetic field: $B = \mu_0 ni$

Energy: $U = \frac{1}{2}Li^2 = \frac{1}{2}(\mu_0 n^2 A\ell)\left(\frac{B}{\mu_0 n}\right)^2 = \frac{B^2}{2\mu_0}A\ell$

Energy density: $u_B = \frac{\text{Energy}}{\text{Volume}} = \frac{U}{A\ell} \rightarrow \boxed{u_B = \frac{1}{2}\frac{B^2}{\mu_0}}$

*** This expression is similar to the energy per unit volume stored in an electric field: $u_E = \frac{1}{2}\varepsilon_0 E^2$

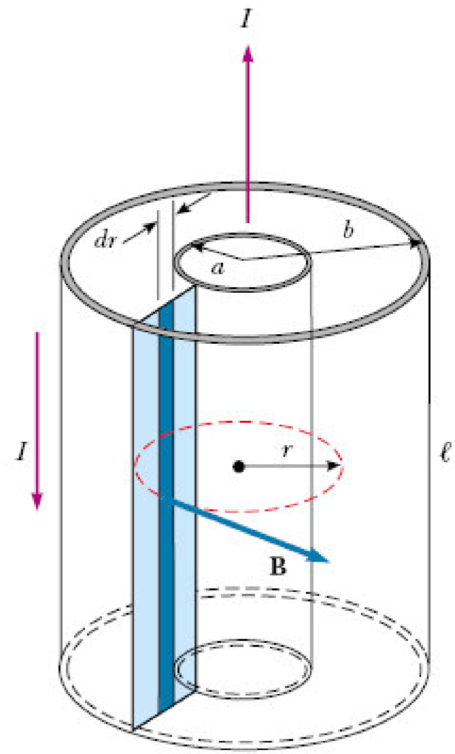
Ques: Where is the energy stored
Ans: Claim (without proof) enegy is stored in magnetic field itself
(Just as in the capacitor / Electric Field sense)

EXAMPLE 6

A long coaxial cable consists of two concentric cylinders with radii a and b, as shown in Figure. Its central conductor carries a steady current *I*, the outer conductor providing the return path.

(a) Calculate the self-inductance of the device

(b) Calculate energy stored in the magnetic field between the conductors.



INFO:
Coaxial cables are often used to connect electrical devices, such as:
your stereo system, antenna cable,
a loudspeaker

SOLUTION

(a) Ampere's Law:

$$\oint \mathbf{B} \cdot d\ell = \mu_0 i_{\text{enc}}$$

$$B(2\pi r) = \mu_0 I$$

or

$$B = \mu_0 I / 2\pi r$$

Area of the strip: $dA = \ell\, dr$

Total flux through entire cross-section:

$$\Phi_B = \int B\, dA = \int_a^b \frac{\mu_0 I}{2\pi r}\ell\, dr = \frac{\mu_0 I \ell}{2\pi}\int_a^b \frac{dr}{r} = \frac{\mu_0 I \ell}{2\pi}\ln\left(\frac{b}{a}\right)$$

Inductance:

$$L = \frac{\Phi_B}{I} = \frac{\mu_0 \ell}{2\pi}\ln\left(\frac{b}{a}\right)$$

(b)

$$U = \tfrac{1}{2}LI^2 = \frac{\mu_0 \ell I^2}{4\pi}\ln\left(\frac{b}{a}\right)$$

Comparison: C* vs *L

	Capacitor	Inductor
Capacitance / Inductance	$C = V / q$	$L = \Phi_B / i$
Energy	$U = q^2 / 2C$	$U = Li^2 / 2$
Energy density	$u_E = \varepsilon_0 E^2 / 2$	$u_B = B^2 / 2\mu_0$
Time constant in a circuit	$\tau = RC$	$\tau = L / R$

EXAMPLE 7

A uniform electric field with a magnitude of 680 kV/m throughout a cylindrical volume results in a total energy of 3.4 μJ. What magnetic field over this same region stores the same total energy?

SOLUTION

$u_E = \frac{1}{2}\varepsilon_0 E^2$ and

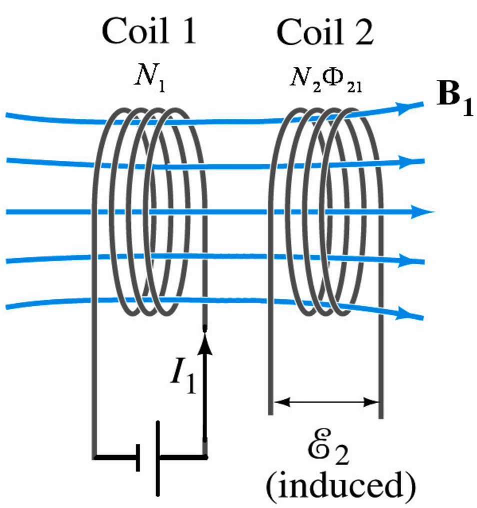
$u_B = \frac{1}{2\mu_0}B^2$ equating them

$\frac{\varepsilon_0 E^2}{2} = \frac{B^2}{2\mu_0}$

solving for B:

$$B = E\sqrt{\varepsilon_0\mu_0} = \frac{6.8\times10^5 \text{ V/m}}{3.0\times10^8 \text{ m/s}} = 2.3\times10^{-3} \text{ T}$$

11-4 Mutual Induction



We define the *mutual inductance* M_{21} of coil 2 with respect to coil 1 as:

$$M_{21} = \frac{N_2\Phi_{21}}{i_1}$$

Recasting:

$$M_{21}\, i_1 = N_2\Phi_{21}$$

If i_1 varies with time:

$$M_{21}\frac{di_1}{dt} = N_2\frac{d\Phi_{21}}{dt}$$

Right hand side of this equation is related to Faraday's law, if we call it emf ε_2 appearing in coil 2:

$$\varepsilon_2 = -M_{21}\frac{di_1}{dt}$$

If the coils are interchanged emf induced on first coil is:

$$\varepsilon_1 = -M_{12}\frac{di_2}{dt}$$

Experiments show that $M_{21} = M_{12}$. Removing subscripts:

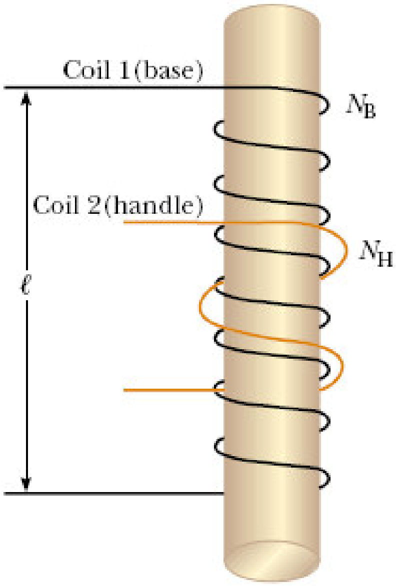
$$M_{21} = M_{12} = M$$

$$\boxed{\varepsilon_2 = -M\frac{di_1}{dt}} \quad \text{and} \quad \boxed{\varepsilon_1 = -M\frac{di_2}{dt}}$$

The induction is indeed mutual. The SI unit for M (as for L) is Henry.

EXAMPLE 8 *Wireless Battery Charger* (Abstracted from Physics, Serway, Beicher)

An electric toothbrush has a base designed to hold the toothbrush handle when not in use. As shown in Figure given below, the handle has a cylindrical hole that fits loosely over a matching cylinder on the base. When the handle is placed on the base, a changing current in a solenoid inside the base cylinder induces a current in a coil inside the handle. This induced current charges the battery in the handle.



We can model the base as a solenoid of length _ with N_B turns (Fig. 32.13b), carrying a source current I, and having a cross-sectional area A. The handle coil contains N_H turns. Find the mutual inductance of the system.

SOLUTION

Magnetic Field : $B = \frac{\mu_0 N_B I}{\ell}$

Mutual inductance : $M = \frac{N_H \Phi_{BH}}{I} = \frac{N_H B A}{I} = \mu_0 \frac{N_H N_B A}{\ell}$

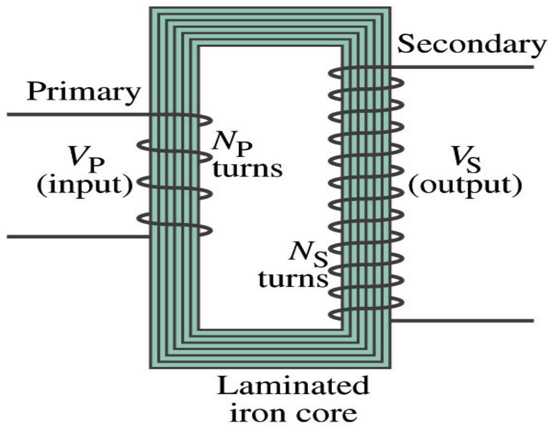
Exercise Calculate the mutual inductance of two solenoids with $N_B = 1\,500$ turns, $A = 1.0 \times 10^{-4}\ \text{m}^2$, $\ell = 0.02$ m, and $N_H = 800$ turns.

Answer 7.5 mH.

11 - 5 Transformers

Mutual Induction is defined as the changing electric current produces a changing magnetic field which can produce a changing current in another conductor. This is the basic principle of the **tansformer.**

A transformer consists of two coils of wires known as primary and secondary. When an AC voltage is applied to the primary, the changing B it produces will induce voltage of the same frequency in the secondary wire.



- In general, it is used by AC voltage for increasing or decreasing an AC voltage
- The two coils can be interwoven or linked by a laminated soft iron core to reduce eddy current losses
- Transformers are designed so that all magnetic flux produced by the primary coil pass through the secondary

So how would we make the voltage different?
By varying the number of loops in each coil.

From Faraday's law, the induced emf in the secondary is $V_S = N_S \frac{d\Phi_B}{dt}$

The input primary voltage is $V_P = N_P \frac{d\Phi_B}{dt}$

Since $d\Phi_B / dt$ is the same, we obtain $\frac{V_S}{V_P} = \frac{N_S}{N_P}$ (transformer equation)

Note that:

- The transformer equation does not work for DC current since there is no change of magnetic flux
- If $N_S > N_P$, the output voltage is greater than the input so it is called a step-up transformer while $N_S < N_P$ is called step-down transformer
- The power output is the same as the input: $V_P I_P = V_S I_S$, therfore:

$$\boxed{\frac{I_S}{I_P} = \frac{V_S}{V_P} = \frac{N_S}{N_P}}$$