Chapter 8:
Particle Systems and Linear Momentum

Up to this point in our study of classical mechanics, we have studied primarily the motion of a single particle or body. To further our comprehension of mechanics we must begin to examine the interactions of many particles at once. To begin this study, we define and examine a new concept, the center of mass, which will allow us to make mechanical calculations for a system of particles.

The Center of Mass of Two Particles

We start by defining and explaining the concept of the center of mass for the simplest possible system of particles, one containing only two particles. From our work in this section we will generalize for systems containing many particles.

Before quantifying our idea of a center of mass, we must explain it conceptually. The concept of the center of mass allows us to describe the movement of a system of particles by the movement of a single point. We will use the center of mass to calculate the kinematics and dynamics of the system as a whole, regardless of the motion of the individual particles.

Center of Mass for Two Particles in One Dimension

If a particle with mass m_1 has a position of x_1 and a particle with mass m_2 has a position of x_2, then the position of the center of mass of the two particles is given by:

$$x_{cm} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}$$

Thus the position of the center of mass is a point in space that is not necessarily part of either particle. This phenomenon makes intuitive sense: connect the two objects with a light but rigid pole. If you hold the pole at the position of the center of mass of the objects, they will balance. That balancing point will often not exist within either object.

Center of Mass for Two Particles beyond One Dimension

Now that we have the position, we extend the concept of the center of mass to velocity and acceleration, and thus give ourselves the tools to describe the motion of a system of particles. Taking a simple time derivative of our expression for x_{cm} we see that:

$$v_{cm} = \frac{m_1v_1 + m_2v_2}{m_1 + m_2}$$

Thus we have a very similar expression for the velocity of the center of mass. Differentiating again, we can generate an expression for acceleration:

$$a_{cm} = \frac{m_1a_1 + m_2a_2}{m_1 + m_2}$$

With this set of three equations we have generated the necessary elements of the kinematics of a system of particles.

From our last equation, however, we can also extend to the dynamics of the center of mass. Consider two mutually interacting particles in a system with no external forces. Let the force exerted on m_2 by m_1 be F_{21}, and the force exerted on m_1 by m_2 by F_{12}. By applying Newton's

Second Law we can state that $F_{12} = m_1a_1$ and $F_{21} = m_2a_2$. We can now substitute this into our expression for the acceleration of the center of mass:

$$a_{cm} = \frac{F_{12} + F_{21}}{m_1 + m_2}$$

However, by Newton's Third Law F_{12} and F_{21} are reactive forces, and $F_{12} = - F_{21}$. Thus $a_{cm} = 0$. Thus, if a system of particles experiences no net external force, the center of mass of the system will move at a constant velocity.

But what if there is a net force? Can we predict how the system will move? Consider again our example of a two body system, with m_1 experiencing an external force of F_1 and m_2 experiencing a force of F_2. We also must continue to take into account the forces between the two particles, F_{21} and F_{12}. By Newton's Second Law:

$$F_1 + F_{12} = m_1a_1$$
$$F_2 + F_{21} = m_2a_2$$

Substituting this expression into our center of mass acceleration equation we get:

$$F_1 + F_2 + F_{12} + F_{21} = m_1a_1 + m_2a_2$$

Again, however, $F_{12} = - F_{21}$, and we can sum the external forces, producing:

$$\sum F_{ext} = m_1a_1 + m_2a_2 = (m_1 + m_2)a_{cm}$$

Let M be the total mass of the system. Thus $M = m_1 + m_2$ and:

$$\sum \boldsymbol{F}_{ext} = M\boldsymbol{a}_{\mathbf{cm}}$$

This equation bears a striking resemblance to Newton's Second Law. In this case, however, we are not speaking of the acceleration of individual particles, but that of the entire system. The overall acceleration of a system of particles, no matter how the individual particles move, can be calculated by this equation. Consider now a single particle of mass M placed at the center of mass of the system. Exposed to the same forces, the single particle will accelerate in the same way as the system would. This leads us to an important statement:
The overall motion of a system of particles can be found by applying Newton's Laws as if the total mass of the system were concentrated at the center of mass, and the external force were applied at this point.

Systems of More than Two Particles

We have derived a method of making mechanical calculations for a system of particles. For simplicity's sake, however, we only derived this for a two- particle system. A derivation for an n particle system would be quite complex. A simple extension of our two particle equations to an n particle system will suffice.

Center of Mass of Many Particles

Previously, M was defined as $M = m_1 + m_2$. However, to continue the study of center of mass we must make this definition more general. If there are n particles in a system, $M = m_1 + m_2 + m_3 + \cdots + m_n$. In other words, M gives the total mass of the system. Equipped with this definition, we can simply state the equations for the position, velocity, and acceleration of the center of mass of a many particle system, similar to the two-particle case. Thus for a system of n particles:

$$x_{cm} = \frac{1}{M} \sum m_n x_n$$

$$v_{cm} = \frac{1}{M} \sum m_n v_n$$
$$a_{cm} = \frac{1}{M} \sum m_n a_n$$

$$\sum F_{ext} = M a_{cm}$$

These equations require little explanation, as they are identical in form to our two particle equations. All these equations for center of mass dynamics may seem confusing, however, so we will discuss a short example to clarify.

Consider a missile composed of four parts, traveling in a parabolic path through the air. At a certain point, an explosive mechanism on the missile breaks it into its four parts, all of which shoot off in various directions, as shown below.

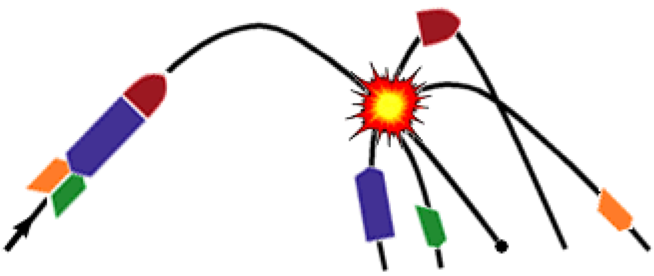

Figure 8.1: A missile breaking into pieces

What can be said about the motion of the system of the four parts? We know that all forces applied to the missile parts upon the explosion were internal forces, and were thus cancelled out by some other reactive force: Newton's Third Law. The only external force that acts upon the system is gravity, and it acts in the same way it did before the explosion. Thus, though the missile pieces fly off in unpredictable directions, we can confidently predict that the center of mass of the four pieces will continue in the same parabolic path it had traveled in before the collision.

Such an example displays the power of the notion of a center of mass. With this concept we can predict emergent behavior of a set of particles traveling in unpredictable ways.
We have now shown a way to calculate the motion of the system of particles as a whole. But to truly explain the motion we must generate a law for how each of the individual particles react. We do so by introducing the concept of linear momentum in the next section.

Impulse and Momentum

Having studied the macroscopic movement of a system of particles, we now turn to the microscopic movement: the movement of individual particles in the system. This movement is determined by forces applied to each particle by the other particles. We shall examine how these forces change the motion of the particles, and generate our second great law of conservation, the conservation of linear momentum.

Impulse

Often in systems of particles, two particles interact by applying a force to each other over a finite period of time, as in a collision. The physics of collisions will be further examined in the next as an extension of our conservation law, but for now we will look at the general case of forces acting over a period of time. We shall define this concept, force applied over a time period, as impulse. Impulse can be defined mathematically, and is denoted by J:

$$\boldsymbol{J} = \boldsymbol{F}\Delta t$$

Just as work was a force over a distance, impulse is force over a time. Work applied mostly to forces that would be considered external in a system of particles: gravity, spring force, friction. Impulse, however, applies mostly to interactions finite in time, best seen in particle interactions. A good example of impulse is the action of hitting a ball with a bat. Though the contact may seem instantaneous, there actually is a short period of time in which the bat exerts a force on the ball. The impulse in this situation is the average force exerted by the bat multiplied by the time the bat and ball were in contact. It is also important to note that impulse is a vector quantity, pointing in the same direction as the force applied.

Given the situation of hitting a ball, can we predict the resultant motion of the ball? Let us analyze our equation for impulse more closely, and convert it to a kinematics expression. We first substitute $\boldsymbol{F} = m\boldsymbol{a}$ into our equation:

$$\boldsymbol{J} = \boldsymbol{F}\Delta t = (m\boldsymbol{a})\Delta t$$

But the acceleration can also be expressed as $\boldsymbol{a} = \frac{\Delta v}{\Delta t}$. Thus:

$$\boldsymbol{J} = m\,\frac{\Delta v}{\Delta t}\,\Delta t = m\Delta \mathbf{v} = \Delta(m\mathbf{v}) = m\mathbf{v_f} - m\mathbf{v_o}$$

The large impulse applied by the bat actually reverses the direction of the ball, causing a large change in velocity.

Recall that when finding that work caused a change in the quantity $\frac{1}{2}mv^2$ we defined this as kinetic energy. Similarly, we define momentum according to our equation for an impulse.

Momentum

From our equation relating impulse and velocity, it is logical to define the momentum of a single particle, denoted by the vector p, as such:

$$\boldsymbol{p} = m\boldsymbol{v}$$

Again, momentum is a vector quantity, pointing in the direction of the velocity of the object. From this definition we can generate two every important equations, the first relating force and acceleration, the second relating impulse and momentum.

Equation 1: Relating Force and Acceleration

The first equation, involving calculus, reverts back to Newton's Laws. If we take a time derivative of our momentum expression we get the following equation:

$$\frac{dp}{dt} = \frac{d}{dt}(mv) = m\frac{dv}{dt} = ma = \sum F$$

Thus

$$\frac{dp}{dt} = \sum F$$

It is this equation, not ***F*** = *m**a*** that Newton originally used to relate force and acceleration. Though in classical mechanics the two equations are equivalent one finds in relativity that only the equation involving momentum is valid, as mass becomes a variable quantity. Though this equation is not essential for classical mechanics, it becomes quite useful in higher-level physics.

Equation 2: The Impulse-Momentum Theorem

The second equation we can generate from our definition of momentum comes from our equations for impulse. Recall that:

$$J = mv_f - mv_o$$

Substituting our expression for momentum, we find that:

$$\boldsymbol{J} = \boldsymbol{p}_\mathbf{f} - \boldsymbol{p}_\mathbf{o} = \Delta\boldsymbol{p}$$

This equation is known as the Impulse-Momentum Theorem. Stated verbally, an impulse given to a particle causes a change in momentum of that particle. Keeping this equation in mind, momentum is conceptually quite similar to kinetic energy. Both quantities are defined based on concepts dealing with force: kinetic energy is defined by work, and momentum is defined by impulse. Just as a net work causes a change in kinetic energy, a net impulse causes a change momentum. In addition, both are related to velocity in some way. In fact, combining the two equations $K = \frac{1}{2}mv^2$ and $p = mv$ we can see that:

$$K = \frac{p^2}{m}$$

This simple equation can be quite convenient for relating the two different concepts.

This section, dealing exclusively with the momentum of a single particle, might seem out of place after a section on systems of particles. However, when we combine the definition of momentum with our knowledge of systems of particles, we can generate a powerful conservation law: the conservation of momentum.

Conservation of Momentum

What happens when a group of particles are all interacting? Qualitatively speaking, each exerts equal and opposite impulses on the other, and though the individual momentum of any given particle might change, the total momentum of the system remains constant. This phenomenon of momentum constancy describes the conservation of linear momentum in a nutshell; in this section we shall prove the existence of the conservation of energy by using what we already know about momentum and systems of particles.

Momentum in a System of Particles

Just as we first defined kinetic energy for a single particle, and then examined the energy of a system, so shall we now turn to the linear momentum of a system of particles. Suppose we have a system of N particles, with masses m_1, m_2,…, m_n. Assuming no mass enters or leaves the system, we define the total momentum of the system as the vector sum of the individual momentum of the particles:

$$\begin{aligned} P &= p_1 + p_2 + \cdots + p_n \\ &= m_1 v_1 + m_2 v_2 + \cdots + m_n v_n \end{aligned}$$

Recall from our discussion of center of mass that:

$$v_{cm} = \frac{1}{M}(m_1 v_1 + m_2 v_2 + \cdots + m_n v_n)$$

where M is the total mass of the system. Comparing these two equations we see that:

$$P = M v_{cm}$$

Thus the total momentum of the system is simply the total mass times the velocity of the center of mass. We can also take a time derivative of the total momentum of the system:

$$\frac{dP}{dt} = M\frac{dv_{cm}}{dt} = Ma_{cm}$$

Recall also that, for a system of particles,

$$\sum F_{ext} = Ma_{cm}$$

Clearly, then:

$$\sum F_{ext} = \frac{dP}{dt}$$

Don't worry if the calculus here is complex. Though our definition of the momentum of a system of particles is important, the derivation of this equation only matters because it tells us a great deal about momentum. When we explore this equation further we will generate our principle of conservation of linear momentum.

Conservation of Linear Momentum

From our last equation we will consider now the special case in which $\sum F_{ext} = 0$. That is, no external forces act upon an isolated system of particles. Such a situation implies that the rate of change of the total momentum of a system does not change, meaning this quantity is constant, and proving the principle of the conservation of linear momentum: *When there is no net external force acting on a system of particles the total momentum of the system is conserved.*

It's that simple. No matter the nature of the interactions that go on within a given system, its total momentum will remain the same. To see exactly how this concept works we shall consider an example.

Conservation of Linear Momentum in Action

Let's consider a cannon firing a cannonball. Initially, both the cannon and the ball are at rest. Because the cannon, the ball, and the explosive are all within the same system of particles, we can thus state that the total momentum of the system is zero. What happens when the cannon is fired? Clearly the cannonball shoots out with considerable velocity, and thus momentum. Because there are no net external forces acting on the system, this momentum must be compensated for by a momentum in the opposite direction as the velocity of the ball. Thus the cannon itself is given a velocity backwards, and total momentum is conserved. This conceptual example accounts for the "kick" associated with firearms. Any time a gun, a cannon, or an artillery piece releases a projectile, it must itself move in the direction opposite the projectile. The heavier the firearm, the slower it moves. This is a simple example of the conservation of linear momentum. By both examining the center of mass of a system of particles, and developing the conservation of linear momentum we can account for a great deal of motion in a system of particles. We now know how to calculate both the motion of the system as a whole, based on external forces applied to the system, and the activity of the particles within the system, based on momentum conservation within the system. This topic, dealing with momentum, is as important as the last one, dealing with energy. Both concepts are universally applied: while Newton's Laws apply only to mechanics, conservation of momentum and energy are used in relativistic and quantum calculations as well.