Chapter 7: Conservation of Energy

Introduction

If the quantity of a subject does not change with time, it means that the quantity is "conserved". The quantity of that subject remains constant even its change between two points is different. The best way to explain that quantity is the energy. If the energy is conserved in a system, then it is known that the total amount of this energy remains constant even its shape changes.

We know that (from previous Chapter) the energy is the capacity to do work. In fact, the work is a kind of energy. The work is related to the force that is applied to a body to change its position. So, there is relation between energy and the force.

Conservative/non-conservative forces

The forces that are applied on bodies to do work are called "conservative" or "non-conservative" forces.

On a straight line, if a body moves forward and backward, then the force exerted on this body is called "conservative". In one dimension, the gravitational force and the spring force can be given as examples for conservative forces.

If a body moves upward, the gravitation force does a work which is negative:

$$\Delta W = -mg(y_f^1 - y_i^1) \tag{7.1}$$

Then, the body falls freely, and the work done by the gravitation force becomes positive:

$$\Delta W = -mg(y_f^2 - y_i^2) \tag{7.2}$$

The total work done by the gravitational force is "0" since we know that the initial and final points for the object is same:

$$y_f^1 = y_i^2, \quad y_f^2 = y_i^1 \quad \Rightarrow \quad W_{net} = \Delta W + \Delta W = 0 \tag{7.3}$$

For the spring force:

$$F = -k\Delta x \tag{7.4}$$

The total work done by the spring is again "0" since $\Delta x_i = x = \Delta x_f$. Then

$$W_{net} = -\frac{1}{2}k(x_f^2 - x_i^2) = 0 \tag{7.5}$$

If a force "F" applied on a body changes its position under a friction force, then this force can not be "conservative". Even though the object changes its position forward and backward, there will be always a friction force which is in the opposite direction. Since the work done by the friction force is negative, the total work done by the force "F" will not be "0". So, the frictional force is a non-conservative force. In summary, a conservative force is a force for which the work done on a closed path is equal to zero.

Conservative systems and mechanical energy

The system means an object and its environment which interacts with that object. A car moving on an inclined plane is in the system of the car itself, the inclined plane, and the earth due to gravitation. So; the system is not only the object but its environment that has effects on the motion of that object. If forces in a system are conservative than this system is called "conservative system".

An example for a conservative system is freely-falling object: If the air resistance is neglected, then the work done on the object is:

$$W = -mg(y_f - y_i) \tag{7.6}$$

Using energy-work theorem,

$$W = K_f - K_i = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 \tag{7.7}$$

and

$$W = -mgy_f + mgy_i = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 \tag{7.8}$$

then

$$mgy_i + \frac{1}{2}mv_i^2 = mgy_f + \frac{1}{2}mv_f^2 \tag{7.9}$$

from this equation we see that each side is equal to a constant. Then we say

$$mgy_i + \frac{1}{2}mv_i^2 = mgy_f + \frac{1}{2}mv_f^2$$

$$U_i + K_i = U_f + K_f \tag{7.10}$$

where U is the potential energy for the object at height "y" and K is the kinetic energy. Then it is seen that

$$U_i + K_i = constant = U_f + K_f$$

$$E_{total} = U + K \qquad (7.10)$$

This is the Mechanical Energy for the object and it is constant.

Potential energy and the conservation of mechanical energy

The amount of change in the potential energy is given by:

$$U = -\int_{x_i}^{x_f} F_x(x)dx = -W = U_f - U_i \tag{7.11}$$

here, the negative sign in the integration shows us that the decrease in the potential is conserved by the increase in the kinetic energy.

$$\Delta U = -W = -(K_f - K_i) = U_f - U_i \tag{7.12}$$

then

$$K_f + U_f = U_i + K_i \tag{7.13}$$

ad we get

$$E_f = E_i \tag{7.14}$$

The energy is conservative in a conservative system.

Gravitational potential energy

The work done by the gravitational force is

$$U_f - U_i = mg(y_f - y_i) \tag{7.15}$$

If $y_i = 0 \quad \Rightarrow \quad U_i = 0, \quad U_f = +mgy_f$

If $y_f = 0 \quad \Rightarrow \quad U_f = 0, \quad U_i = -mgy_i$

There is no difference for the sign of the potential energy for the object. You can take whether $y_f = 0$ or $y_i = 0$. The difference between two points will be always equal to each other.

Potential energy for a spring

The Hooke law states that

$$F(x) = -k\,x \tag{7.16}$$

and the work done by the spring is given by

$$W = -\frac{1}{2}k(x_f^2 - x_i^2)\,. \tag{7.17}$$

Since the potential energy is

$$\Delta U = -W \quad \Rightarrow \quad U_f - U_i = \frac{1}{2}k(x_f^2 - x_i^2) \tag{7.18}$$

If $x_i = 0 \quad \Rightarrow \quad U = \frac{1}{2}kx^2$

As seen in the last Equation, the potential energy for a spring can not be “negative”.

Non-conservative forces

We know that the mechanical energy is written as

$$K_f - K_i = W_{net} \tag{7.19}$$

Now, we know that the work done by an object may not be conservative all time. Then we write the net work done by the force as

$$W_{net} = W_{cons} + W_{noncons} \tag{7.20}$$

and we know $W_{cons} = -(U_f - U_i)$, so

$$W_{net} = -(U_f - U_i) + W_{noncons} \tag{7.21}$$

and we said

$$K_f - K_i = -(U_f - U_i) + W_{noncons}$$

$$K_f + U_f = K_i + U_i + W_{noncons}$$

$$E_f = E_i + W_{noncons} \tag{7.22}$$

here, it is seen obviously that $E_f - E_i = 0$ and $W_{noncons}$ can not be calculated easily. To find $W_{noncons}$ we must know the initial and final conditions of the object.

Examples and Problems

Question 7.1:
As it is known the air resistance is a kind of force with a magnitude proportional to v^2, and it always acts in the opposite direction of the velocity of the particle. Is it a conservative force? Explain your reason.

Solution 7.1:
No! Let us consider an object thrown into the air: it first reaches a maximum height then returns to the ground. It thus completes a round trip (closed path). By our first principle of conservative forces, the total work done by air resistance over this closed loop must be zero. However, since the air resistance always opposes the motion of objects moving with a velocity, it acts in the opposite direction as the displacement of the object for the entire trip. Thus the total work over this closed loop must be negative, and air resistance, much like friction, is a non-conservative force.

Question 7.2:
The force of a mass on a spring is given by $F(x) = -kx$. Calculate the total work done by the spring over one complete oscillation. To do this calculation, assume that the initial displacement of the mass is from *d*, to -*d*, and then it returns back to its original displacement from –*d,* to *d* (In this way confirm the fact that the spring force is conservative!).

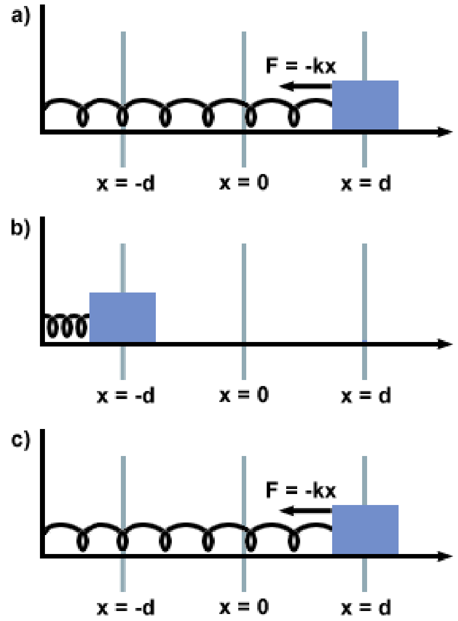

Figure 7.1: a) initial position of mass, b) position of mass halfway through oscillation, c) final position of mass.

Solution 7.2:

To calculate the total work done during the trip, we must evaluate the integral $W = \int_{x_i}^{x_f} F(x)dx$. Since the mass changes its directions in the motion, we must actually evaluate two integrals: one from *d,* to -*d*, and one from –*d,* to *d*:

$$W = \int_{x_i}^{x_f} F(x)dx = \int_{d}^{-d} (-kx)dx + \int_{-d}^{d} (-kx)dx$$

$$W = \left[-\frac{1}{2}kx^2\right]_d^{-d} + \left[-\frac{1}{2}kx^2\right]_{-d}^{d} = 0 + 0$$

and finally
$W = 0$

Thus the total work done over a complete oscillation (a closed loop) is zero, confirming that the spring force is indeed conservative.