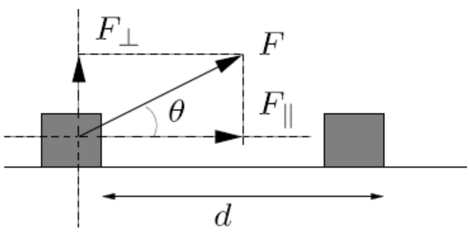
Chapter 6: Work and Energy

Introduction

So far we have just learned everything about Newtonian Physics: Kinematics in which the motion itself is investigated and Dynamics in which the reasons and affects of the motion is investigated. What we are going to study next is derived from what we have learned. However, the results we will obtain are more refined, and possibly will need a deeper conceptual understanding.

The Work done by a constant force

We give the definition of the concept of work in physics, and worry about the reason and motivation later. In a simplest way of saying it, the work in physics is a product of the displacement of an object and the parallel component of the force applied on that object as it moves. Let us consider an example to illustrate what we mean: A box is pulled by a force *F*, which makes an angle θ with the horizontal, to move a distance *d* to the right. What is the work done by the force? We can see from the diagram that the force *F* is not in the same direction of the displacement. So we can decompose the force *F* into perpendicular and parallel components:



$$F_{||} = F\cos(\theta)\,, \qquad \text{and} \qquad F_\perp = F\sin(\theta) \tag{6.1}$$

But the work done by this force on the object is given by

$$W = F_{||}d = F\,d\cos(\theta) \tag{6.2}$$

It is clearly seen that if $\theta < 90^\circ$, the work is positive, in which case, the force contributes the effect of speeding up the object; if $\theta > 90^\circ$, the work is negative, and the force contributes the effect of slowing down the object; the work is zero when $\theta = 90^\circ$.

It can be concluded that the condition for a force on an object to do non-zero work:

a) The object has to move. While it is a great workout to hold still some heavy weight, you do not do any work from the point of view of physics. (Most people think that it takes a lot of work to <u>hold</u> a *250kg* weight in the air: The weight is not moving, though, so in the sense of physics no work is done. It is important to realize how our definition differs from a common one, and stick to the physical understanding of work.)

b) The force has not to be perpendicular to the motion. In the above example, the normal force and gravity is perpendicular to the (horizontal) displacement, so they contribute no work.

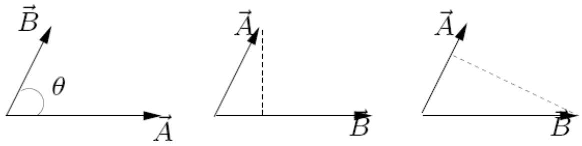
It should be noted also that when we talk about *F = ma*, *F* has to be a net (or total) force acting on the object. About work, on the other hand, we can talk about the work done by an individual force.

Dot product of two vectors

We know that the work is scalar and it is a certain product of the force and the displacement. We know that both force and displacement are vectors. We have also learned some things about the rules of vectors in the previous Chapters and now let us discuss the product of two vectors. Work is a scalar and then it is a scalar product of two vectors:

$$W = \vec{F} \bullet \vec{d} = |\vec{F}||\vec{d}|\cos(\theta) \tag{6.3}$$

where θ is the angle between the two vectors. As we mentioned before, the ***scalar product*** is also called ***dot product***, since conventionally, we use a dot to denote the product.



At this point, we will discuss some properties of the scalar product: Let us consider two vectors $\vec{A}$ and $\vec{B}$ shown in the figure. We know that the dot product of two vectors can be defined as the projection of one vector ($\vec{A}$) on the other vector ($\vec{B}$), or vice versa. We know that

$$\vec{A} \bullet \vec{B} = |\vec{A}||\vec{B}|\cos(\theta)$$

$$\vec{A} \bullet \vec{A} = A^2$$
$$\vec{A} \bullet \vec{B} = \vec{B} \bullet \vec{A}$$
$$\vec{A} \bullet (\vec{B} + \vec{C}) = \vec{A} \bullet \vec{B} + \vec{A} \bullet \vec{C}) \tag{6.4}$$

and then we can write

$$\vec{A}_{||} = |\vec{A}|\cos(\theta) \quad ,and \quad \vec{B}_{||} = |\vec{B}|\cos(\theta)$$

$$\vec{A} \bullet \vec{B} = \vec{A}_{||}\vec{B} = \vec{B}_{||}\vec{A} \tag{6.5}$$

This implies us that the work done by a force can be stated in two equivalent ways:

- **a.** Work done by a force is equal to the product of the displacement and the parallel component of the force along the displacement.

- **b.** Work done by a force is equal to the product of the force and the parallel component of the displacement along the direction of the force.

The Scalar (dot) product can be done if the vector components are given: If the vectors $\vec{A}$ and $\vec{B}$ are given as

$$\vec{A} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k} \quad and \quad \vec{B} = b_x\hat{i} + b_y\hat{j} + b_z\hat{k} \tag{6.6}$$

then the result is given by

$$\vec{A} \bullet \vec{B} = a_x b_x + a_y b_y + a_z b_z \tag{6.7}$$

where $\hat{i} \bullet \hat{i} = \hat{j} \bullet \hat{j} = \hat{k} \bullet \hat{k} = 1$.

The Work done by a non-constant force

In the previous section, the force is assumed to be constant. Unfortunately, forces are not constant in general. So, how does one compute the work? As we do always, we divide the whole into the parts: We can slice the displacement into small segments, in each small segment the force can be viewed as a constant, so the work done by the force in such a small segment is given by

$$\Delta W \approx \vec{F} \bullet \Delta\vec{r} = F\ \Delta r\,. \tag{6.8}$$

For one-dimension, it can be written as

$$\Delta W \approx \vec{F} \bullet \Delta\vec{x} = F\ \Delta x\,. \tag{6.9}$$

This is the work done in this infinitesimal segment. The force will be applied to the object and there will be an initial point where the motion starts and a final point where the motion ends. If we sum these infinitesimal works, then

$$W \approx \sum F\Delta x\,. \tag{6.10}$$

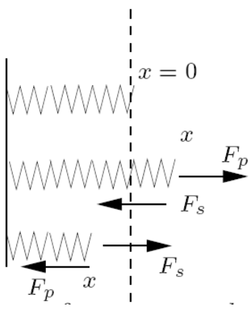
As $\Delta x \to 0$, the summation will be an integral between points where the motion starts and ends:

$$W \approx \lim_{\Delta x \to 0} \sum F\Delta x = \int_{x_1}^{x_2} F dx\,. \tag{6.11}$$

So, the work done by a non-constant force that is applied to object between the points x_1 and x_2 is given by

$$W = \int_{x_1}^{x_2} F(x)\, dx\,. \tag{6.12}$$

It should be noticed that the force is x-dependent.



As we know that the simplest varying force is the spring force, as shown in the figure. We observe that the more you stretch the spring, the more force you need. If we don't stretch it too much to break it, the force we need to stretch a spring is

$$F = kx\,. \tag{6.13}$$

where *k* is a constant (spring constant), and *x* is the amount stretched or compressed from its normal length. By the action and reaction, the spring we exert a force to pull back the spring into its normal length, which is given by

$$F_s = -kx\,. \tag{6.14}$$

the minus sign here implies that the direction of the force is always the opposite to the displacement *x*.

Work done by a spring force from *x* to its normal position (*x=0*) is given by

$$W = \int_{x_1}^{x_2} F(x)\, dx = \int_0^x -kx\, dx = -k\int_0^x x\, dx = -k\left(\frac{x^2}{2}\right]_0^x = -\frac{1}{2}kx^2\,. \tag{6.15}$$

Here, the minus sign indicates that the displacement of the mass attached to the spring is opposite to the spring force. In fact, the best way to determine the sign is to inspect the situation. If the spring force is in the same direction as the displacement, then the work is positive; if the spring force is in the opposite direction as the displacement, then the work is negative.

The general formula for the work done by a non-constant force is given as line integral in three dimensions by using the same procedure we have done through Equation (6.8) to Equation (6.11):

$$\Delta W \approx \vec{F} \bullet \Delta\vec{r}$$
$$\Delta W = (F_x\hat{i} + F_y\hat{j} + F_z\hat{k}) \bullet (dx\hat{i} + dy\hat{j} + dz\hat{k})$$
$$W = \int_i^f \left(F_x dx + F_y dy + F_z dz\right) \tag{6.16}$$

Work-Energy relation

How do we use work? What is its significant in the mechanics? We derive an equation in which a relation is obtained between work and energy, especially kinetic energy.

We know the Newton second law that defines the acceleration of a body:

$$\vec{a}_x = \frac{\sum \vec{F}_x}{m} \tag{6.17}$$

This is the one-dimensional acceleration of a body. If the object moves with a constant acceleration from x_i to x_f with the velocities v_i to v_f, then we can write

$$v_f^2 - v_i^2 = 2a_x(x_f - x_i) \tag{6.18}$$

and

$$a_x(x_f - x_i) = \frac{1}{2}v_f^2 - \frac{1}{2}v_i^2 \tag{6.19}$$

Let us calculate the net work done on the object by the net force applied on it:

$$W_{net} = \left(\sum F_x\right)\left(x_f - x_i\right), \tag{6.20}$$

inserting the net force term into this equation, we get

$$W_{net} = \left(ma_x\right)\left(x_f - x_i\right) = m\left[a_x\left(x_f - x_i\right)\right] = m\left(\frac{1}{2}v_f^2 - \frac{1}{2}v_i^2\right)$$
$$W_{net} = m\frac{1}{2}v_f^2 - m\frac{1}{2}v_i^2. \tag{6.21}$$

If we define $K = m\frac{1}{2}v^2$, then we get

$$W_{net} = K_f - K_i\ . \tag{6.22}$$

This is the work-energy theorem: The work done by the net force on an object is equal the change of its kinetic energy.

We should note that in this derivation, we only used the Newton's 2nd law, and some mathematics. We did not make any assumption on the properties of the forces, thus the work-energy principle above can be universally applied. In fact, the work-energy is correct even in the area where the Newtonian physics is no longer valid.

The Power

The "Power" is the speed of the work done. As we know the Mechanical Systems, an engine for example, are not limited by the amount of work they can do, but rather by the rate at which they can perform the work. So; this quantity, the rate at which work is done, is defined as power. From this very simple definition, we can come up with a simple equation for the average power of a system. If the system does an amount of work, *W*, over a period of time, *T*, then the average power is simply given by:

$$\overline{P} = \frac{W}{t} \tag{6.23}$$

It is important to remember that this equation gives *the average power* over a given time, not *the instantaneous power*. Again we should remember, because in the equation of the work, *W*, increases with *x*, even if a constant force is exerted, the work done by the force increases with displacement, meaning the power is not constant. To find *the instantaneous power*, we must use some calculus again: The instantaneous power means that the time interval for the work done becomes zero, then we should differentiate the work with respect to time:

$$P = \frac{dW}{dt} \tag{6.24}$$

In the sense of this second equation for power, *power is the rate of change of the work done by the system*. From this equation, we can derive another equation for instantaneous power that does not rely on calculus. Given a force that acts at an angle θ to the displacement of the particle,

$$P = \frac{dW}{dt} \quad \Rightarrow \quad P = \frac{d\left[\vec{F} \bullet \vec{x}\right]}{dt} \quad \Rightarrow \quad P = \frac{d\left[F\ x\cos(\theta)\right]}{dt}$$

$$P = \frac{d\left[F\ x\cos(\theta)\right]}{dt} = F\cos(\theta)\frac{dx}{dt} \quad \Rightarrow P = \quad F\cos(\theta)v \tag{6.24}$$

In the last term of the Equation (6.24), it is obviously seen that we can the power equation as

$$P = \quad F\cos(\theta)v \quad \Rightarrow \quad P = \quad \vec{F} \bullet \vec{v} \tag{6.25}$$

Though the calculus is not necessarily important to remember, the final equation is quite valuable. We now have two simple, numerical equations for both the average and instantaneous power of a system. It should be noted that, in analyzing this equation, we can see that if the force is parallel to the velocity of the particle, then the power delivered is simply $P = Fv$.

Up to now, we have just mentioned about the equations for the power: what about its unit? The unit of power is *the joule per second* which is more commonly called a "***Watt***" (Watt is the surname of *James Watt* (1735-1819) who has spent his life for studying on the steam machines). Another unit commonly used to measure power, especially in everyday situations, is "***the horsepower***", which is equivalent to about ***746 Watts***. As we know in our daily life, the rate at which our automobiles do work is measured in horsepower. Power, unlike work or energy, is not really a "building block" for further studies in physics. We do not derive other concepts from our understanding of power. It is far more applicable for practical use with machinery that delivers force. That said, power remains an important and useful concept in classical mechanics, and often comes up in physics courses.