Chapter 4: Motion in Two Dimensions

Part-1

In this lesson we will discuss motion in two dimensions. In two dimensions, it is necessary to use vector notation to describe physical quantities with both magnitude and direction. In this chapter, we will begin by defining ***displacement***, ***velocity*** and ***acceleration*** as vectors in two dimensions. Then, we will discuss the solution of projectile motion problems in two dimensions, such as the motion of a cannon fired at a target at an angle, the motion of a cliff diver jumped straight off or the motion of a nuclear bomb dropped from a fighter at a height.

In the first section, some definitions are given. In the second section, derivations for the equations of motion in two-dimensions are shown. The equations for the uniform speed circular motion and the non uniform speed circular motion are derived in the third section. In the last section, the relative motion in two dimensions is contained. Analytical and numerical examples are solved at the end of each section.

Displacement, Velocity and Acceleration in 2-Dimensions

As we mentioned in 1-dimension, the vector nature of velocity and acceleration is taken into account by the sign (positive or negative) of the quantity. In 2-dimensions we must use 2 components to specify a velocity or acceleration vector. That is the merely difference, in equations, which may be enough to make something difficult!

If there is a vector lying in the X-Y plane, it can be written as a component in the X-direction added to a component in the Y-direction. Let $\vec{A}$ be a two-component vector in the X-Y plane. Then it is written as $\vec{A} = A_x\hat{i} + A_y\hat{j}$ where A_x and A_y are the X and Y components of the vector $\vec{A}$ (See Figure 4.1a-b).

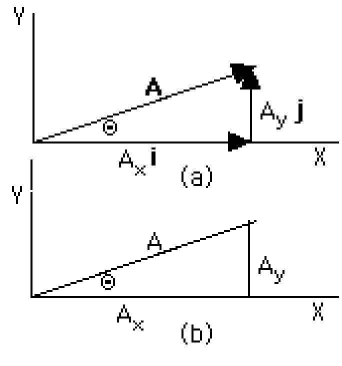

Figure 4.1: Components of a vector

From Fig. 1b, we see that $A_x = A\cos(\Theta)$ and $A_y = A\sin(\Theta)$, and also $|\vec{A}| = \sqrt{A_x^2 + A_y^2}$

If there two vectors, there will be a third one whose magnitude and direction is found by the vector operation on the others. Let us call $\vec{C}$ be the resultant vector of the addition of $\vec{A}$ with $\vec{B}$. Then, it is written as;

$\vec{A} = A_x\hat{i} + A_y\hat{j}$ and $\vec{B} = B_x\hat{i} + B_y\hat{j}$

$\vec{C} = \vec{A} + \vec{B}$

$\vec{C} = (A_x + B_x)\hat{i} + (A_y + B_y)\hat{j}$

$\vec{C} = C_x\hat{i} + C_y\hat{j}$

where $C_x = A_x + B_x$ and $C_y = A_y + B_y$. See Figure 4.2 for the representation.

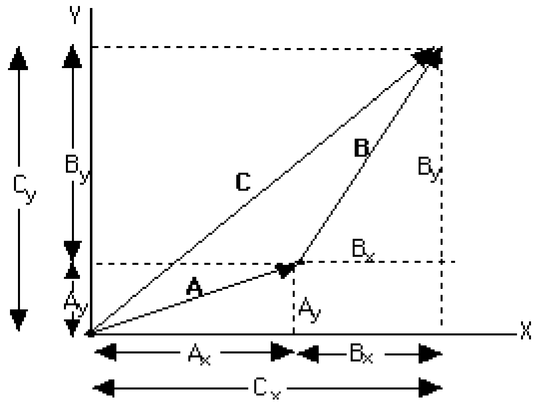

Figure 4.2: Vector addition, resultant vector and their components

It is seen also in Figure 4.2 that the resultant vector represents change in position of an object from point A to point B. So, the resultant vector is a displacement vector of an object that moves from A to B.

Displacement

In Figure 4.3, an object is initially at position $\vec{r}_i(t_i)$ at time t_i (point A). Some time later, t_f, the object is at position $\vec{r}_f(t_f)$ (point B). The displacement vector of the object is given by:

$$\Delta\vec{r} = \vec{r}_f - \vec{r}_i \tag{4.1}$$

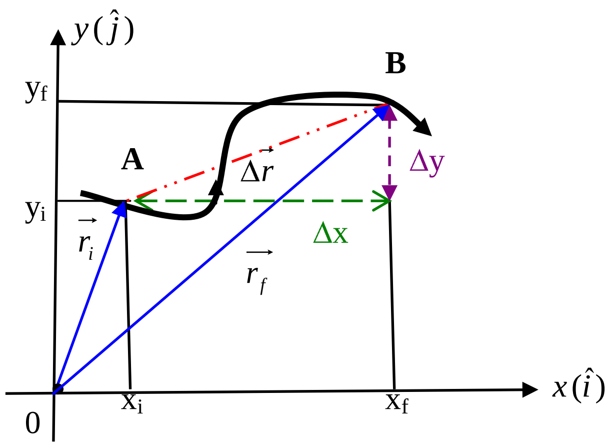

Figure 4.3: Displacement vector

Average Velocity
Using the result of displacement, we can find the average velocity of the object between time intervals:

$$\vec{v}_{av} = \frac{\Delta\vec{r}}{\Delta t} = \left(= \frac{Total\ Displacement}{Elapsed\ Time} \right) = \frac{\vec{r_f} - \vec{r_i}}{t_f - t_i} = \frac{\vec{r_f} - \vec{r_0}}{t_f - t_0} \tag{4.2}$$

Generally, the initial conditions are assumed to be at "0" point. We will use this notation for the initial condition after that point. As with the 1-dimensional definition, average velocity is independent of the path between the end points.

Instantaneous Velocity
As it is mentioned in the previous chapter, the instantaneous velocity is given by

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta\vec{r}}{\Delta t} = \frac{d\vec{r}}{dt} \tag{4.3}$$

Average Acceleration
It is the change in velocity over the change in time:

$$\overrightarrow{a_{av}} = \frac{\Delta\vec{v}}{\Delta t} = \frac{\overrightarrow{v_f} - \overrightarrow{v_0}}{t_f - t_0} \tag{4.4}$$

The direction of the acceleration is in the direction of the vector Δ**v,** and its magnitude is $|\Delta$**v**$/\Delta$t $|$.

Instantaneous Acceleration
As we have followed before, **Instantaneous acceleration** is calculated by taking shorter and shorter time intervals, i.e. when $\Delta t \to 0$, then;

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta\vec{v}}{\Delta t} = \frac{d\vec{v}}{dt} \tag{4.5}$$

Note: a particle can accelerate in different ways:

1. The magnitude of $\vec{v}$ can change in time, while the direction of motion stays the same.
2. The magnitude of $\vec{v}$, $|\vec{v}|$, can stay constant, while the direction of motion changes. This only happens in more than one dimension.
3. Both $|\vec{v}|$ and the direction of $\vec{v}$ can change.

Motion in 2D with Uniform (Constant) Acceleration

We know that

$$\overrightarrow{a_{av}} = \frac{\Delta\vec{v}}{\Delta t} = \frac{\overrightarrow{v_f} - \overrightarrow{v_0}}{t_f - t_0} = \frac{\vec{v}(t_f) - \vec{v}(t_0)}{t_f - t_0}. \tag{4.6}$$

In case of uniform acceleration, the average acceleration would be equal to the instantaneous acceleration. Since the definition of average acceleration is as above, then the instantaneous acceleration may be written as

$$\vec{a} = \frac{\vec{v}(t_f) - \vec{v}(t_0)}{t_f - t_0} \tag{4.7}$$

From that equation, we can write the velocity equation as

$$\vec{v}(t_f) = \vec{v}(t_0) + \vec{a} \times (t_f - t_0). \tag{4.8}$$

As the motion is in 2-dimensional space, the unit vectors along x and y-axis will be $\hat{i}$ and $\hat{j}$, respectively. So taking the rectangular components of acceleration (even it is uniform!) and the initial velocity, we get

$$\vec{v}(t_f) = \left(v_x + a_x \times (t_f - t_0)\right)\hat{i} + \left(v_y + a_y \times (t_f - t_0)\right)\hat{j}. \tag{4.9}$$

This is the equation of velocity of an object with uniform acceleration in 2-dimensional motion. Note that the sub-indices "*x*" and "*y*" show the initial values of the parameters along the x-axis and y-axis.

The position vector of the object is written by using Eq. (4.2),

$$\vec{r}(t_f) = \vec{r}(t_0) + \vec{v}_{av} \times (t_f - t_0) \tag{4.10}$$

Since the velocity of the object increases uniformly, then we can write it

$$\vec{v}_{av} = \frac{1}{2}\left(\vec{v}(t_f) + \vec{v}(t_0)\right) \tag{4.11}$$

as we have done in the previous Chapter. Replacing this result into the Eq. (4.11), then we get the position vector of the object in terms of velocity and acceleration:

$$\begin{aligned} \vec{r}(t_f) &= \vec{r}(t_0) + \vec{v}_{av} \times (t_f - t_0) \\ \vec{r}(t_f) &= \vec{r}(t_0) + \left[\frac{1}{2}\left(\vec{v}(t_f) + \vec{v}(t_0)\right)\right] \times (t_f - t_0) \end{aligned} \tag{4.12}$$

and since the final velocity is given as in Eq. (4.8), then the result for the position vector in terms of the initial values can be given as

$$\begin{aligned} \vec{r}(t_f) &= \vec{r}(t_0) + \left[\frac{1}{2}\left(\vec{v}(t_f) + \vec{v}(t_0)\right)\right] \times (t_f - t_0) \\ \vec{r}(t_f) &= \vec{r}(t_0) + \left[\frac{1}{2}\left(\vec{v}(t_0) + \vec{a} \times (t_f - t_0) + \vec{v}(t_0)\right)\right] \times (t_f - t_0) \\ \vec{r}(t_f) &= \vec{r}(t_0) + \vec{v}(t_0) \times (t_f - t_0) + \frac{1}{2} \times \vec{a} \times (t_f - t_0)^2. \end{aligned} \tag{4.13}$$

It is noted that all variables are in 2-dimension. For simplicity this equation may be written as follows by comparing the coefficient of $\hat{i}$ and $\hat{j}$. Since

$$\vec{r}(t_f) = \vec{x}(t_f) + \vec{y}(t_f) \tag{4.14}$$

then

$$\begin{aligned} \vec{x}(t_f) &= \vec{x}(t_0) + \vec{v}_x \times (t_f - t_0) + \frac{1}{2} \times \vec{a}_x \times (t_f - t_0)^2 \\ \vec{y}(t_f) &= \vec{y}(t_0) + \vec{v}_y \times (t_f - t_0) + \frac{1}{2} \times \vec{a}_y \times (t_f - t_0)^2 \end{aligned} \tag{4.15}$$

where the sub-indices "x" and "y" show the initial values of the parameters along the x-axis and y-axis, again.

Projectile Motion

As we know well, the projectile motion is a particular kind of 2 dimensional motion. Firstly, we will make the following assumptions:

- The only force present is the force due to gravity.
- The magnitude of the acceleration due to gravity is $|g| = g = 9.8m/s^2$. We choose a coordinate system in which the positive y-axis points up perpendicular to the earth's surface. This definition gives us that $\overrightarrow{a_y} = -g(\hat{j}) = -9.8m/s^2$ and $\overrightarrow{a_x} = 0$.
- The rotation of the earth does not affect the motion.

Initial Conditions:
We choose the coordinate system so that the particle leaves the origin ($x_0 = 0,\ y_0 = 0$) at time $t_i = 0$ with an initial velocity of $\vec{v}_i$. The Procedure for Solving Projectile Motion Problems are as followings:

1. We will separate the motion into the x (horizontal) part and y (vertical) part.

2. Then we will consider each part separately using the appropriate equations. The equations of motion, for each component, become:
   a. x-motion (a_x =0);
   It is seen from Eq. (4.9) that the x-component of the velocity will only have initial value. So,

$$\vec{v}_x = v_{0x}\,\hat{i}$$
$$\vec{v}_x = v_0\cos(\theta)\,\hat{i} \quad (const.) \tag{4.16a}$$
$$\vec{x}(t) = v_{0x}t\,\hat{i}$$
$$\vec{x}(t) = v_0\cos(\theta)t\,\hat{i} \tag{4.16b}$$

   b. y-motion ($\overrightarrow{a_y} = -g(\hat{j}) = -9.8m/s^2$);
   It is seen that only the gravitational force is applied on the motion of the body ; then the y-component of the velocity will be given as

$$\vec{v}_y(t) = v_{0y}\hat{j} - gt\,\hat{j}$$
$$\vec{v}_y(t) = v_0\sin(\theta)\hat{j} - gt\,\hat{j} \tag{4.17a}$$
$$\vec{y}(t) = y_0\,\hat{j} + v_{0y}t\ \ \hat{j} - \frac{1}{2}gt^2\,\hat{j}$$
$$\vec{y}(t) = y_0\,\hat{j} + v_0\sin(\theta)t\ \ \hat{j} - \frac{1}{2}gt^2\,\hat{j} \tag{4.17b}$$

We can also write the velocity equation that is time-independent (as we have done in previous Chapter.)

$$v_y^2(t) = v_{0y}^2 - 2gy(t) \tag{4.18}$$

Note that this result is obviously just a magnitude. The direction of the velocity may be decided by the values of the last term, position vector-y(t)-, of the equation.

3. Finally, we will solve the resulting system of equations for the unknown quantities.

It is also worth to practice on the results obtained above. If the time variable in Eq. (4.16b) is inserted into the Eq. (4.17b), then it is seen that the position equation of the object becomes as parabolic equation.

$$y = \tan(\theta)\, x - \left(\frac{g}{2v^2 \cos^2(\theta)} \right) x^2 \tag{4.19}$$

It means that the path of the object in the projectile motion is parabolic:

$$y = a + b\,x + c\,x^2 \tag{4.20}$$

In the projectile motion, there will be a final distance on the x-axis the object arrives. If the Eq. (4.19) is solved for x when y=0, then the range of projectile is found that

$$R = \frac{v_0^2 \sin(2\theta)}{g} \tag{4.21}$$

It is obvious that $\theta = 45^\circ$ for the max range, $R_{\max}$.

The max height that the object reaches can be found by using the Eq. (4.18). Since the vertical velocity will be "0" when the object reaches the highest height, then

$$0 = v_{0y}^2 - 2gy(t), \text{ and } \vec{v}_{0y} = v_0 \sin(\theta)\,\hat{j}, \text{ then}$$

$$0 = (v_0 \sin^2(\theta)) - 2gy_{\max}, \text{ finally}$$

$$y_{\max} = \frac{v_0^2 \sin^2(\theta)}{2g} \tag{4.22}$$

is found, see Figure.

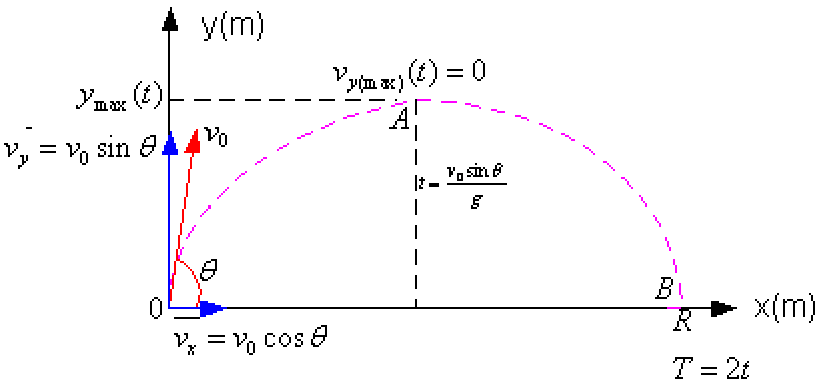

Figure 4.4: The projectile motion

The total time for all these motion is found by using Eq. (4.17a). Since the velocity of the object is zero at a time "t" when it reaches at the highest point but the horizontal velocity has some magnitude, then we can write

$$\vec{v}_y(t) = v_{0y}\,\hat{j} - gt\,\hat{j} \quad \Rightarrow \quad 0 = v_0 \sin(\theta)\,\hat{j} - gt\,\hat{j} \tag{4.23}$$

then

$$t = \frac{v_0 \sin(\theta)}{g} \tag{4.24}$$

This is the time expression for the object that has reached the highest point, *A*, where the vertical component of the velocity is "zero". It is seen that it takes some time for that object falling from point *A* to *B* since the projectile does a parabolic path. So, the total time for the projectile to reach from *0* to *B* in a parabolic path is

$$T = 2t = \frac{2v_0 \sin(\theta)}{g} \tag{4.25}$$

Examples and Problems

Question 4.1:
A bombardment aircraft having velocity of 180mi/h leaves its bomb with 30^0 angle downward in horizontal line. The horizontal distance between the point the bomb leaved and the point where it hits the ground is 701m.

a) Find the height at which the aircraft leaves the bomb and
b) Find the flight time of the bomb.

Solution 4.1:
We are given:

$$v_0 = 180mi/h = 80.5m/s, \quad \theta = 30^0, \quad x_0 = 701m,$$

The point where the bomb is leaved is assumed to be "0" point. Then

$$y_0 = 0, \quad y = y_0 + v_0 \sin(\theta)t + \frac{1}{2}at^2 \quad \Rightarrow \quad y = y_0\hat{j} - v_0 \sin(\theta)t\hat{j} - \frac{1}{2}gt^2\hat{j}$$

and

$$x_0 = v_0 \cos(\theta)ti \quad \Rightarrow \quad t = \frac{x_0}{v_0 \cos(\theta)}$$

If we replace "t" in the equation of height, then

$$-y\hat{j} = 0 - v_0 \sin(\theta)t\hat{j} - \frac{1}{2}gt^2\hat{j}, \quad then$$

$$-y\hat{j} = 0 - v_0 \sin(\theta)\left(\frac{x_0}{v_0 \cos(\theta)}\right)\hat{j} - \frac{1}{2}g\left(\frac{x_0}{v_0 \cos(\theta)}\right)^2\hat{j}$$

The solution of the is equation for the unknown, y , gives us that

$$y = -900\hat{j}$$

This means that the bomb goes downward vertically. To find the time for the bomb to reach the ground is

$$t = \frac{x_0}{v_0 \cos(\theta)} \quad \Rightarrow \quad t = \frac{701m}{80.5m/s\cos(30)} \quad \Rightarrow \quad t = 10.15s$$

Question 4.2:
Assume that your young sister swings on a rope above the local swimming hole on a hot day in her summer holiday (See Figure 4.5). She lets go of the rope when her initial velocity is 2.05 m/s at an angle of 35.0° above the horizontal. If her flight in air takes for 1.10 s, how high above the water was she when she let go of the rope?

Solution 4.2:
We are given:

$v_0 = 2.05 m/s \qquad \theta = 35^0 \qquad t_{Flight} = 1.10 s$

Firstly, we should find the velocity vectors in x and y- directions:

$\overrightarrow{v_{0y}} = v_0 \sin(\theta)\hat{j}$

$\overrightarrow{v_{0y}} = 2.05\sin(35) = 1.176 m/s(\hat{j})$

$\overrightarrow{x_{0y}} = v_0 \cos(\theta)\hat{i}$

$\overrightarrow{x_{0y}} = 2.05\cos(35)\hat{i} = 1.679 m/s(\hat{i})$

At the end of the flight time, she will enter into the water.

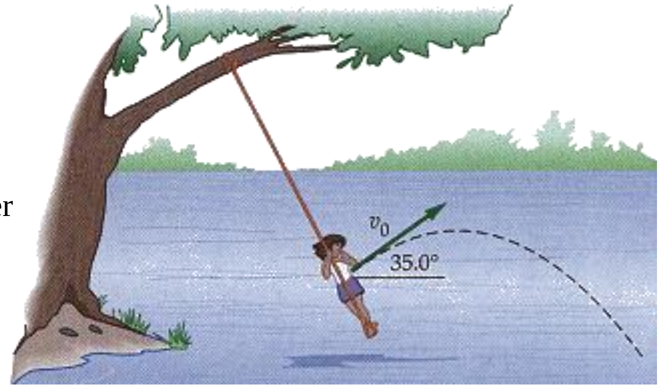

Figure 4.5: Swinging girl on a rope above water

Since the final height is "0" then the total displacement in the vertical position is just

$$-y\hat{j} = y_0\hat{j} + v_0 \sin(\theta)t\hat{j} - \frac{1}{2}gt^2\hat{j}$$

then

$$0 = -y_0\hat{j} + v_0 \sin(\theta)t_{Flight}\hat{j} - \frac{1}{2}gt_{Flight}{}^2\hat{j}$$

Where the velocity vector is taken "+" since the motion is 35^0 upward initially. Then

$$y_0\hat{j} = v_{0y}t_{Flight}\hat{j} - \frac{1}{2}gt_{Flight}{}^2\hat{j} \quad \Rightarrow \quad y_0\hat{j} = (1.176 m/s)(1.10 s)\hat{j} - \frac{1}{2}(9.8 m/s^2)(1.10 s)^2\hat{j}$$

then the initial height is found as

$y_0\hat{j} = -4.641 m(\hat{j})$

It should be noted that the result means that the motion is in (-) vertical direction!

Part-2

Circular Motion in Two Dimensions (Uniform Speed)

A particle moving along a circular path constant speed is said to be in uniform circular motion. As the particle moves around the circle, its angular position on the circle changes. So being a tangent at each position, the velocity vector is perpendicular to the position vector ***r***. Since the speed of the particle is constant (for the first case) so that the magnitude of the velocity is constant; but the direction of the velocity vector is changing from one position to another position as time goes. Therefore such type of motion has an acceleration whose magnitude remains constant but direction changes from one position to another one. This acceleration is called "*centrifugal acceleration*". The difference between centripetal and centrifugal accelerations is quite simple - *centrifugal* forces do not exist while *centripetal* accelerations do. As with most simple statements, there is a great deal more to understanding this issue than simply memorizing which of the accelerations does or does not exist. To understand the centripetal acceleration and the fictitious centrifugal acceleration, let's first examine the words *centripetal* and *centrifugal*.

- centri is derived from the Latin *centr* meaning "center."
- petal is derived from the Latin *petere* meaning "seek."
- fugal is derived from the Latin *fugere* meaning "to flee" as in fugitive.

So, literally, the centripetal acceleration is a "center-seeking" force. The fictitious centrifugal acceleration is, literally, a non-existing "center-fleeing" acceleration.

Let us impose on a point-mass object the condition that it is on a circular path at any time. The vector velocity of this object is always tangent to the circle therefore it changes direction in time, as the object moves along the circle. Consequently, the circular motion is a accelerated motion, simply because the direction of vector velocity changes, even if its magnitude (speed) remains constant. The problem now becomes to find out what force generates the acceleration that keeps the object moving on a circle.

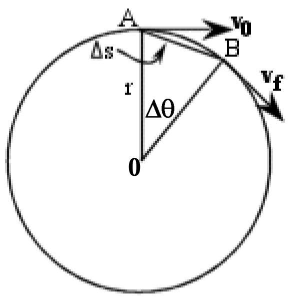

Figure 6: Diagram of a point-mass object moving along a circular path.

Consider an initial object position A and a position B where the object reaches after a time interval Δt (Figure 6). The velocity vectors at A and B are shown in the Figure as $\mathbf{v_0}$ and $\mathbf{v_f}$, respectively. The arc of circle traveled by the object in the time interval Δt is called Δs. Force simplicity, in this example, the magnitude of velocity (speed) to be constant. Such a motion of an object on a circle, with constant speed (v), is called "uniform circular motion". Again, such a motion is an "accelerated motion" just because the direction of the vector velocity changes. Since the object displaces its position with time, its displacement changes by an amount of $\Delta\theta$ in Δt time. The rate of change of this angular displacement with respect to time is given by

$$\vec{w} = \frac{\Delta\theta}{\Delta t} \tag{4.26}$$

and this change is called "**angular velocity**". For the limit condition;

$$\vec{w} = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt} \quad \Rightarrow \quad \vec{w} = \dot{\theta}\,. \tag{4.27}$$

Since the angle changes from *0* to 2π in time *T*, then the angular velocity can be written as

$$w = \frac{2\pi}{T}.$$

where the T is the "***period***" of the particle that takes time for "*1 revolution*". So, the revolution for 1 second is written as

$$f = \frac{1}{T}.$$

This is the "***frequency***" of the particle orbiting around a center. Then the angular velocity can be written as

$$w = 2\pi f \ .$$

The linear velocity of the object is written as,

$$\overrightarrow{\Delta v} = \frac{\overrightarrow{\Delta s}}{\Delta t} \quad \Rightarrow \quad \overrightarrow{\Delta s} = \overrightarrow{\Delta \theta}\, r = \overrightarrow{\Delta w}\, \Delta t\, r \quad \Rightarrow \overrightarrow{\Delta v} = \frac{\overrightarrow{\Delta w}\, \Delta t\, r}{\Delta t} \ . \tag{4.28}$$

$$\overrightarrow{\Delta v} = r\, \overrightarrow{\Delta w}$$

and so that the relation between the angular velocity and the linear velocity is given by the equation

$$|v| = |wr|. \tag{4.29}$$

It should be noted that the linear velocity does not change in "*r*" direction; it only changes in angular positions. Then it is concluded that the linear velocity has a magnitude of "*wr*" but its direction vector in "*r*" remains constant. Because of this, there is just magnitude-relation between *w* and *v*.

Since the acceleration, **a**, is the change in velocity over time:

$$\vec{a} = \frac{\overrightarrow{\Delta v}}{\Delta t} \tag{4.30}$$

We notice that acceleration is a vector, **Δv**, multiplied by a scalar, 1/Δt. So, the direction of the acceleration will be the same as the direction of the change in velocity, **Δv**. So what is the vector "change of velocity" **Δv**?
The "change" in any physical quantity is defined as the final quantity minus the initial quantity. So, the change in velocity is the final vector velocity minus the initial vector velocity:

$$\overrightarrow{\Delta v} = \overrightarrow{v_f} - \overrightarrow{v_0} \tag{4.31}$$

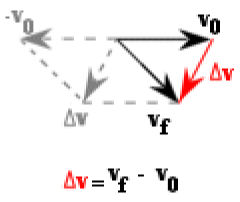

Figure 7: Evaluation of $\overrightarrow{\Delta v} = \overrightarrow{v_f} - \overrightarrow{v_0}$

To find the vector **Δv** we graphically subtract vector $\mathbf{v_0}$ from vector $\mathbf{v_f}$ (Figure 7). Note: for clarity, we have moved the points of origin of both vectors $\mathbf{v_f}$ and $\mathbf{v_0}$ to a common point, out of Figure 6. (Remember, since a vector is defined only by magnitude and direction, its point of origin is irrelevant.) Figure 7 shows the vector **Δv** in red color. We can now transport it back into the initial Figure (Figure 6), preserving its magnitude and direction as given in Figure 7. After executing this operation, we call the Figure 8. So, we can draw in Figure 8 all three vectors (**a** - blue, **F**- green, and **Δv** - red) parallel to each other (Here, the "F" is the force that is the subject of next Chapter.)

It should be noticed the direction of all these three vectors; they are all directed toward the center of the circle and are, therefore, "center-seeking" or centripetal.

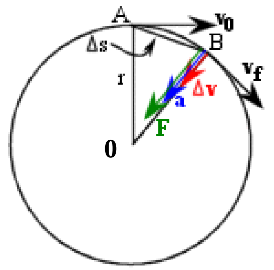

Figure 8: Same as Figure 6, showing the vectors **F**, **a**, and **Δv**.

<u>Derivation of an Analytic Expression for "a"</u>

Now, in order to derive an analytic expression for "**a**", we use the following property of two "similar" isosceles triangles, as show in Figure 9:

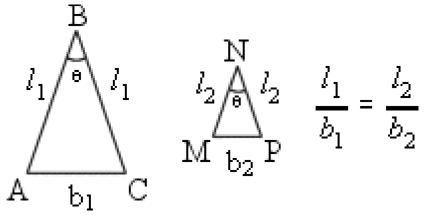

Figure 9: The relation between the sides of "similar" isosceles triangles.

For the triangles in Figure 6 and Figure 7, this relationship is:

$$\frac{r}{\Delta s} = \frac{v}{\Delta v} \tag{4.32}$$

from which:

$$\Delta v = \frac{v}{r}\Delta s \tag{4.33}$$

Divide this equation by the time interval Δt:

$$\frac{\Delta v}{\Delta t} = \frac{v}{r}\left(\frac{\Delta s}{\Delta t}\right) \tag{4.34}$$

then

$$a_r = \frac{v}{r}v \quad \Rightarrow \quad a_r = \frac{v^2}{r} \tag{4.35}$$

This expression represents the magnitude of a_r (centripetal acceleration). If we want to express them as vectors, it becomes:

$$\vec{a}_r = -\left(\frac{v^2}{r}\right)\hat{r} \tag{4.36}$$

where $\hat{\mathbf{r}}$(r-hat) represents the "unit vector" along r and it is obvious that the acceleration has minus sign in direction. The centripetal acceleration is along the inward radius. It can be seen in Figure 7 that as **Δv** becomes smaller and smaller then its direction becomes inward radius (it directs into the origin "0"!).

In the Cartesian coordinates, the variables can be understood more easily as seen in Figure 10.

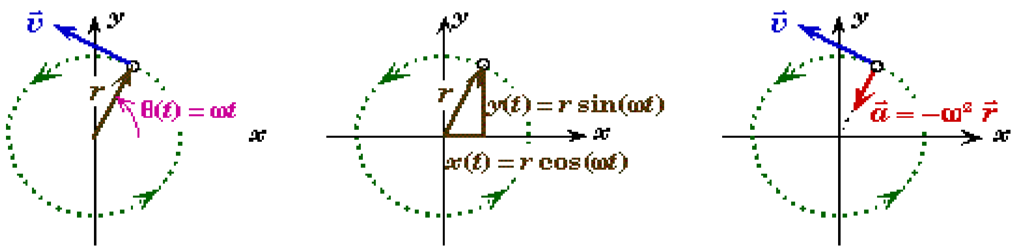

Figure 10: Positional vector diagram of a point-mass object moving along a circular path and its x and y- axis components.

The location of the object at any moment – that means the equation of motion - relative to the center of rotation is given by

$$\vec{r}(t) = r_x\hat{i} + r_y\hat{j} = r\cos(\theta)\hat{i} + r\sin(\theta)\hat{j} \qquad (4.37)$$

using the Equation (4.26), then

$$\vec{r}(t) = r\cos(wt)\hat{i} + r\sin(wt)\hat{j} \qquad (4.38)$$

where i and j, with the little hats, are the unit vectors in the x and y-directions. The Eq. (4.38) is the equation of motion for the body in the Cartesian coordinates.

The object's velocity is easily found by taking the derivative of its location with respect to time:

$$\vec{v}(t) = \frac{d\vec{r}(t)}{dt} = \frac{d}{dt}\left(r\cos(wt)\hat{i} + r\sin(wt)\hat{j}\right)$$

$$\vec{v}(t) = -rw\sin(wt)\hat{i} + rw\cos(wt)\hat{j} \qquad (4.39)$$

This velocity is always tangent to the circle or equivalently, $\vec{v}(t)$ is always perpendicular to position vector, ***r***, and $\vec{v}(t) \bullet \vec{r} = 0$

The object's acceleration is easily found by taking the derivative of its velocity with respect to time:

$$\vec{a}(t) = \frac{d\vec{v}(t)}{dt} = \frac{d}{dt}\left(rw\sin(wt)\hat{i} + rw\cos(wt)\hat{j}\right)$$

$$\vec{a}(t) = -rw^2\cos(wt)\hat{i} - rw^2\sin(wt)\hat{j}$$

$$\vec{a}(t) = -w^2\left(r\cos(wt)\hat{i} + r\sin(wt)\hat{j}\right)$$

$$\vec{a}(t) = -w^2\,\vec{r}(t) \qquad (4.40)$$

It is obviously seen that the direction of the object's acceleration $\vec{a}(t)$ is opposite $\vec{r}$, i.e., $\vec{a}(t)$ directed towards the center of motion.

Examples and Problems

Question 4.3:
What is the centripetal acceleration of the satellite orbiting at 640km above the Equator if it takes 98min for one revolution around the earth? (r_{earth}=6370km) (Note that the result will the "g" at this height!)

Solution 4.3:
We are given:

$$h = 640km \quad , \quad r_{earth} = 6370km$$

$$t = 98\min = 8580s$$

Since the radius at which the satellite orbits is the distance of the satellite to the center of the earth, then the total distance of it to the orbiting center is

$$r = h + r_{earth} = 640km + 6370km \quad \Rightarrow \quad r = 7010km = 7.01\times 10^6\,m$$

then the acceleration of the satellite is

$$a = \frac{v^2}{t} = \frac{\left(\frac{2\pi r}{t}\right)^2}{t} = \frac{4\pi^2 r^2}{t^2}$$

$$a = \frac{4\pi^2 (7.01\times 10^6 m)^2}{(5880s)^2} = 8.0 m/s^2$$

Question 4.4:

An object orbiting uniformly around a center, having radius of 1km, spreads 1^0 angle in 0.1s.

a. What is the linear velocity of this object?

b. What is the acceleration of the object?

c. Find the position vector in x and y components of that object for t=1s.

Solution 4.4:

We are given:

$\Delta\theta = 1^0, \quad \Delta t = 0.1s, \quad r = 1km$

Since the object displaces 1degree in 0.1second, then

$$w = \frac{\Delta\theta}{\Delta t} = \frac{1^0}{0.1} \qquad w = \frac{(1^0 * 2\pi / 360^0)}{0.1} = \frac{2\pi}{36} rad/\sec$$

Then the linear velocity

$$v = wr = 1km * \frac{2\pi}{36} rad/\sec$$

$$v = \frac{2000\pi}{36} = 174.45 m/s$$

The acceleration is

$$a = \frac{v^2}{r} = \frac{174.45^2}{1000} \quad \Rightarrow \quad a = 30.43 m/s^2$$

The position vector is given by

$$\vec{r}(t) = r\cos(wt)\hat{i} + r\sin(wt)\hat{j}$$

then

$$\vec{r}(t) = 1000\cos(\frac{2\pi}{36}t)\hat{i} + 1000\sin(\frac{2\pi}{36}t)\hat{j}.$$

For the 1second, the position vector components are:

$$\vec{r}(t=1) = 1000\cos(\frac{2\pi}{36}*1)\hat{i} + 1000\sin(\frac{2\pi}{36}*1)\hat{j}.$$

$$\vec{r}(t=1) = 984.81\hat{i} + 173.65\hat{j}$$

So,

x=984.81m and y=173.65m