Chapter 3: Motion in One Dimension

In this lesson we will discuss motion in one dimension. The oldest one of the Physics subjects is the ***Mechanics*** that investigates the motion of a body. It deals not only with a football, but also with the path of a spacecraft that goes from the Earth to the Mars! The Mechanics can be divided into two parts: ***Kinematics*** and ***Dynamics***.

The kinematics aims the motion. It is important for the kinematics that which the path body follows. It answers the questions such that: Where the motion started? Where the motion stopped? What time has taken for the complete of motion? What the velocity body had?.

The dynamics deals with the effects that create the motion or change the motion or stop the motion. It takes into account the forces and the properties of the body that can affect the motion.

After that point, we will enter into the world of kinematics, first. The One-Dimensional Motion is the starting point for the kinematics. We will introduce some definitions like ***displacement, velocity*** and ***acceleration***, and derive equations of motion for bodies moving in one-dimension with ***constant acceleration***. We will also apply these equations to the situation of a body moving under the influence of gravity alone.

As Leonardo Da Vinci said, "*To understand motion is to understand nature"*, we need to understand the motion by observing and doing practice (experiment) on it, first. The reason is quite simple. Those things that are of interest in science are the things that undergo change. It is basic principle that: to understand how something works, you have to see it in action. The workings of the universe include anything in the universe which experiences change according to some repetitive pattern. It is fact that change could be in the form of a chemical reaction, an increase or decrease in the population of butterflies, etc. In *nature*, the easiest changes to observe are those of motion: An object is moved from one position in space to another. In fact, the motion generally leaves the object itself unchanged and thus simplifies the observation.

We first need some definitions to identify the motion. It begins by defining the change in position of a particle. We call it "displacement".

Displacement is defined to be the change in position or distance that an object has moved and is given by the equation;

$$\Delta\vec{r} = \vec{r_2} - \vec{r_1} \tag{3.1}$$

where $\vec{r_2}$ is the final position and $\vec{r_1}$ is the initial position. The arrow indicates that displacement is a vector quantity: it has direction and magnitude, as we mentioned before. In fact, $\vec{r}$ is in 3-dimension and written as $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$. In our calculations, we will only take x-component of **r** into account. In 1-dimension, there are only two possible directions that can be specified with either a plus (+) or a minus (-) sign. As we know, other examples of vectors are *velocity*, *acceleration* and *force*. In contrast, scalar quantities have only magnitude. Some examples of scalars are *speed*, *mass*, *temperature* and *energy*.

It is not enough to define the displacement for a motion. In order to be useful, we also need to specify something about ***time***. After all, a motion that causes a displacement of 1 meter can be large or small depending on whether it took a second or a thousand years to do it. The international standard unit of time is *the second*. Hence, time and space are inextricably linked in physics since we need both to explain motion and motion is fundamental to all areas of physics. In the language of mathematics, we describe the changes in position, $\Delta\vec{x} = \vec{x_f} - \vec{x_i}$ or time as

$$\Delta t = t_f - t_i \tag{3.2}$$

where the *i* and *f* subscripts depict **initial** and **final**, respectively. Generally, $t_i = t_0 = 0$.

There is no doubt that this use of symbols requires some patience if you are not already comfortable with it. The normal tendency of students new to physics is to immediately replace symbols with numbers as soon as possible. Experience will show you that this is generally not a good thing to do. What's most important to you is the very practical problem of mistakes: you tend to make more algebra errors with numbers than with symbols (despite what your instincts may tell you, this is *always* true!). What's most important to a scientist, however, is that the symbols represent the essence. The numbers are hardly ever the important point in understanding what's going on. Numbers can be changed by situation, choice, or any number of unexplained reasons, but the mathematical description of what happens to the symbols is what represents the underlying truth. In other words, if you derive an equation by mathematical rules that correctly describe the way nature works it is the equation that is always true. The numbers that go into the equation can vary by large amounts, but whatever their values, they must always satisfy the equation! Our hope for the course is to make the language of the equations second nature to you so that the essence of the science represented by them is clear. Not for "now", but also for "future".

If we continue our rather legal-sounding definitions, it is noted that there are two other aspects of motion that are important. The first is that we would like to quantify the *amount* of motion taking place. Therefore, we should define the ***average velocity*** or *speed* of an object. As we said above, the idea of quantifying motion involves both the distance traveled by the body and the time the body took to travel it. For that reason, it makes since to define the average velocity that is the displacement over total time, $\vec{v}_{av}$, as follows:

$$\vec{v}_{av} = \frac{\Delta\vec{x}}{\Delta t} = \left(= \frac{Total\ Displacement}{Elapsed\ Time} \right) = \frac{\vec{x_f} - \vec{x_i}}{t_f - t_i} = \frac{\vec{x_f} - \vec{x_i}}{t} \tag{3.3}$$

This equation has the right characteristics for what we desire: if the time interval is small or the displacement is large, we have a large value for $\vec{v}_{av}$, i.e. a lot of motion is taking place.

On the other hand, if the time interval is large or the displacement is small, then the average velocity is small, i.e. not much is happening as far as motion is concerned. Notice that on the graphs like the ones we showed above, the average velocity is just the slope of the **x** vs. **t** data. **Note:** Δt is always > 0 so the sign of depends only on the sign of Δx.

Graphical interpretation of velocity:
Consider 1-dimensional motion from point A (with coordinates x_i , t_i) to point B (at x_f , t_f). We can plot the trajectory on a graph (see Figure 3.1).

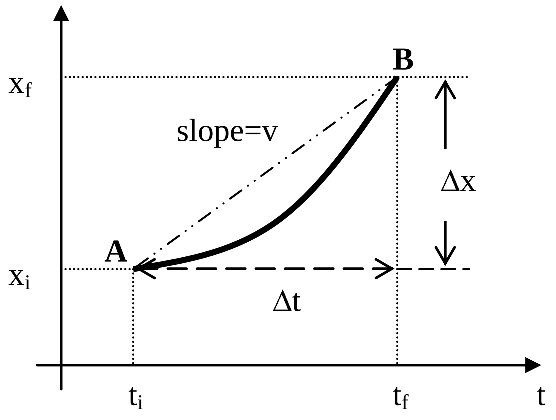

Figure 3.1: Graphical interpretation of velocity

Then $\vec{v}_{av}$ from Eq. (3.3) is just the slope of the line joining A and B. We have to notice that we only deal with the final and initial point. We know "nothing" about the motion of the body that moves between point A and point B. We do not know what path the body followed OR what the shape of the body is OR what kinds of forces are applied on the body OR the body applied on surroundings? It means that you wake up in the dormitory and have gone to class in the morning and returned to the dormitory after 8 hours. Since the total displacement is "ZERO", your average velocity is "ZERO". ☺

Let's assume that the average velocity of the particle be different for different time intervals. In that condition, the particle's velocity will differ for each time interval and it will be necessary to calculate its velocity for a given certain time, t. This leads us to define ***instantaneous velocity***. Since the particle may not follow a straight line on its path, the displacement vectors of that particle will differ from each one by direction and magnitude. If the displacement occurs in time Δt after a certain time t, then the displacement will be Δ**r.** If the displacement is infinitesimal small, then the velocity will take a certain value for that time interval. In Mathematics,

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta \vec{x}}{\Delta t} = \frac{d\vec{x}}{dt} \qquad (3.4)$$

This is the ***instantaneous velocity*** of the particle and its magnitude is called ***speed***.

Average acceleration is the change in velocity over the change in time:

$$\overrightarrow{a_{av}} = \frac{\Delta \vec{v}}{\Delta t} = \frac{\overrightarrow{v_f} - \overrightarrow{v_i}}{t_f - t_i} = \frac{\overrightarrow{v_f} - \overrightarrow{v_i}}{t} \qquad (3.5)$$

The direction of the acceleration is in the direction of the vector Δ**v,** and its magnitude is |Δ**v**/Δt |.

As we have followed before, we can find the **Instantaneous acceleration** of the body; **Instantaneous acceleration** is calculated by taking shorter and shorter time intervals, i.e. taking $\Delta t \to 0$:

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt} \tag{3.6}$$

It should be noted that:
Acceleration is the rate of change of velocity.
When velocity and acceleration are in the same direction, speed increases with time.
When velocity and acceleration are in opposite directions, speed decreases with time.
Graphical interpretation of acceleration: On a graph of *v* versus *t*, the average acceleration between A and B is the slope of the line between A and B, and the instantaneous acceleration at A is the tangent to the curve at A.

From now on “**velocity**” and “**acceleration**” will refer to the **instantaneous** quantities.

One Dimensional Motion with Constant Acceleration

As it is understood from the subtitle, constant acceleration means velocity increases or decreases at the same rate throughout the motion. Example: an object falling near the earth's surface (neglecting air resistance).

Derivation of Kinematics Equations of Motion

We choose $t_i = 0, \quad x_i = x_0, \quad v_i = v_{0x}$ and $t_f = t, \quad x_f = x, \quad v_f = v_x$. Since $\vec{a}$ =constant, then $\vec{a} = a$. Then Eq. (3.5) can be written as $a_x = \frac{v_x - v_{0x}}{t}$, or

$$v_x = v_{0x} + a_x\, t \tag{3.7}$$

Since “a ” is constant, v_x changes uniformly and $\bar{v}_x = \frac{1}{2}(v_{0x} + v_x)$. From Eq. (3.3), we know that $\bar{v}_x = \frac{x - x_0}{t}$. Now, we can combine $x - x_0 = \bar{v}_x\, t = \frac{1}{2}(v_{0x} + v_x)\, t$ and can use Eq. (3.7) we get:

$$\begin{aligned} x - x_0 &= \bar{v}_x\, t = \frac{1}{2}(v_{0x} + v_x)\, t = \frac{1}{2}(v_{0x} + v_{0x} + a_x)\, t \\ x &= x_0 + v_{0x} t + \frac{1}{2} a_x t^2 \end{aligned} \tag{3.8}$$

This is the equation of motion for a particle in 1-dimension, with a constant acceleration.

If we rewrite Eq. (3.7), we find $v_x = v_{0x} + a_x t \quad \Rightarrow \quad t = (v_x - v_{0x})/a_x$. Then substituting this result into the last term of Eq. (3.8) we find

$$v_x^{\,2} = v_{0x}^2 + 2a_x(x - x_0) \tag{3.9}$$

It should be noted that this equation does not depend on time, *t*.

Freely Falling Bodies

We call that a **freely falling object** is an object that moves under the influence of gravity only. By neglecting air resistance, all objects in free fall in the earth's gravitational field have a constant acceleration that is directed towards the earth's center, or perpendicular to the earth's surface, and of magnitude $\vec{a} = g = 9.81 m/s^2$. If motion is straight up and down and we can choose a coordinate system with the positive y-axis pointing up and perpendicular to the earth's surface, then we can describe the motion with Eq. (3.7), Eq. (3.8), Eq. (3.9) with $a \rightarrow -g, \quad x \rightarrow y$. (Negative sign arises because the coordinate system is changed and the acceleration direction is downward.)

So that, equations of motion for the 1-dimensional vertical motion of an object in free-fall can be written as following:

$$\begin{aligned} v_y &= v_{0y} - g\,t \\ y &= y_{0y} + v_{0y}t - \frac{1}{2}gt^2 \\ v_y^2 &= v_{0y}^2 - 2\,g\,y \end{aligned} \tag{3.10}$$

Note: Since the acceleration due to gravity is the same for any object, a heavy object does not fall faster than a light object.

One Dimensional Motion with Variable Acceleration

The procedure is the same as we have done in the previous section. The velocity vector is

$$\begin{aligned} \vec{v} &= \frac{d\vec{r}}{dt} = \frac{d}{dt}\left[x\hat{i} + y\hat{j} + z\hat{k}\right] \\ \vec{v} &= \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k} \end{aligned} \tag{3.11}$$

Then the acceleration is given as

$$\vec{a} = \frac{d\vec{v}}{dt}. \tag{3.12}$$

Since the acceleration is variable, then it can't be written as a constant. We should write the equation with respect to the velocity vector components, taking derivation of them. So that,

$$\vec{a} = \frac{d}{dt}(v_x\hat{i} + v_y\hat{j} + v_z\hat{k})$$
$$\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k} \tag{3.13}$$

In the one-dimensional system, we can write the equation;

$$\vec{a} = a_x\hat{i} \tag{3.14}$$

Examples and Problems

Question 3.1:
Assume that a car decelerates at $2.0m/s^2$ and comes to a stop after traveling 25m.

a) Find the speed of the car at the start of the deceleration and
b) Find the time required to come to a stop.

Solution 3.1:
We are given:

$a = -2.0m/s^2$

$x = 25m$

$v = ?$

a) From

$v^2 = v_0^2 + 2ax \quad \Rightarrow \quad v_0^2 = v^2 - 2ax \quad \Rightarrow \quad v_0^2 = 0 - 2(-2)(25) = 100$

$v_0 = 10m/s$

b) From $v = v_0 + at$ we have $t = \dfrac{v - v_0}{a} = \dfrac{-10}{-2} = 5s.$

Question 3.2:
Assume that a car traveling at a constant speed of 30m/s passes a police car at rest. The policeman starts to move at the moment the speeder passes his car and accelerates at a constant rate of $3.0m/s^2$ until he pulls even with the speeding car.

a) Find the time required for the policeman to catch the speeder and
b) Find the distance traveled during the chase.

Solution 3.2:
We are given, for the speeder:

$v_0^s = 30m/s$, constant speed, then $a^s = 0$

and for the policeman:

$a_0^p = 3.0m/s^2$

a) The distance traveled by the speeder is given as $x^s = v^s t = 30t$. Distance traveled by policeman $x^p = x_0^p + v_0^p t + \frac{1}{2}a_p t^2$. When the policeman catches the speeder $x^s = x^p$ or,

$$x^s = x^s \quad \Rightarrow \quad 30t = 0 + 0 + \frac{3t^2}{2},$$

Solving for t we have $t = 0$ and $t = \frac{2}{3}(30) = 20s$. The first solution tells us that the speeder and the policeman started at the same point at $t = 0$, and the second one tells us that it takes 20 s for the policeman to catch up to the speeder.

b) Substituting back in above we find the distance that the speeder has taken

$$x^s = 30(20) = 600m$$

And also for the policeman

$$x^p = x_0^p + v_0^p t + \frac{1}{2}a_p t^2 = 0 + 0 + \frac{1}{2}(3)(20)^2 = 600m$$

Question 3.3:
A rocket moves upward, starting from rest with an acceleration of 29.4m/s^2 for 4 s. At the end of this time, it runs out of fuel and continues to move upward. How high does it go totally?
Solution 3.3:
For the first stage of the flight we are given:

$$a = 29.4m/s^2 \quad for \quad t = 4s$$

This gives us the velocity and position at the end of the first stage of the flight:

$$v_1 = v_0 + at = 0 + 29.4(4) = 117.6m/s$$

and

$$y_1 = y_0 + v_o t + \frac{1}{2}at^2 = 0 + 0 + \frac{1}{2}(29.4)(4)^2 = 235.2m$$

For the second stage of the flight, the rocket will go upward with its velocity till it stops (That is the Newton's Motion Law).So, we start with

$$v_1 = 117.6m/s$$

$$a = g = -9.8m/s^2$$

And we end up with $v_2 = 0$. We want to find the distance traveled in the second stage ($y_2 - y_1$). We have,

$$v_2^2 = v_1^2 + 2a(\Delta y) \quad \Rightarrow \quad \Delta y = y_2 - y_1 = \frac{v_2^2 - v_1^2}{2g} = \frac{0^2 - (117.6)^2}{2(-9.8)} = 705.6m.$$

Therefore, the total distance taken by the rocket is $y_2 = y_1 + 705.6m = 235.2 + 705.6 = 940.8m$

Question 3.4:
A train with a constant speed of 60km/h goes east for 40min. Then it goes 45° north-east for 20min. And finally it goes west for 50min. What is the average velocity of the train?
Solution 3.4:
We are given, for the train:

$$v_0 = 60km/h = 1km/\min$$

$t_1 = 40\min, east$

$t_2 = 20\min, 45^\circ\ north-east$

$t_3 = 50\min, west$

If we write the position vectors for the train for each motion:

$\overrightarrow{d_1} = (v_0 t_1)\hat{i} = (1*40)\hat{i} = 40\hat{i}$

$\overrightarrow{d_2} = (v_0 t_2 \cos\theta)\hat{i} + (v_0 t_2 \sin\theta)\hat{j} = 10\sqrt{2}(\hat{i} + \hat{j})$,

$\overrightarrow{d_3} = (v_0 t_3)(-\hat{i}) = -50\hat{i}$

The average velocity is given by;

$$\vec{v}_{average} = \frac{Total\ Displacement}{Elapsed\ Time} = \frac{\Delta\vec{d}}{\Delta t} = \frac{\overrightarrow{d_1} + \overrightarrow{d_2} + \overrightarrow{d_3}}{\Delta t}$$

$$\vec{v}_{average} = \frac{40\hat{i} + 10\sqrt{2}(\hat{i} + \hat{j}) + (-50\hat{i})}{40 + 20 + 50} = \frac{10(\sqrt{2} - 1)\hat{i} + 10\sqrt{2}\hat{j}}{110} km/\min$$

$$\left|\vec{v}_{average}\right| = \sqrt{\frac{(\sqrt{2} - 1)^2}{11} + \frac{(\sqrt{2})^2}{11}} = 0.13 km/\min = 8.0 km/h$$

$$\alpha = \tan^{-1}\left(\frac{\sqrt{2}/11}{(\sqrt{2} - 1)/11}\right) = 73.7^\circ$$

α is the angle between the final vector and east direction.

Question 3.5:

The motion of a particle is given by $x = t^2 + 3t - 3$, where x is distance in meter and t is time in sec.

a) Find the velocity of the particle after 10 sec.

b) Find also acceleration of the particle. State whether acceleration is uniform of variable.

Solution 3.5:

We are given, for the particle:

$x = t^2 + 3t - 3$,equation of motion for a particle in 1-D.

To find the velocity of the particle at a certain time, we should take the differentiation of the equation of motion:

$$v = \frac{dx}{dt} = \frac{d}{dt}[t^2 + 3t - 3],$$

$v = 2t + 3$

We find the velocity equation of that particle. Then, for the 10second,

$v(t = 10s) = 2(10) + 3 = 23 m/s$,

This is the velocity of the particle at time t=10seconds.

To find the acceleration of the particle, we differentiate the velocity w.r.t. time t, then

$$a = \frac{dv}{dt} = \frac{d}{dt}[2t + 3],$$

$a = 2 m/s^2$

In final equation, it is seen that the acceleration does not depend on time and has constant value of $a = 2 m/s^2$. Therefore the acceleration of the particle is uniform (constant) and not variable.