Chapter 2: Vectors

In this lesson we will examine some of the elementary ideas concerning vectors. The reason for this introduction to vectors is that many concepts in science, for example, displacement, velocity, force, acceleration, have a size or magnitude, but also they have associated with them the idea of a direction. And it is obviously more convenient to represent both quantities by just one symbol. That is the **vector**.

A change of position of a particle is called *displacement*. The displacement of particle is given by some numerical values and the direction of it. For example, "She has gone 50m north." That means she has moved 50m and her motion was towards north. "The hurricane destroyed the city with a speed of 190km/h." means that we know hurricane has 190km/h speed (not velocity!) but we do not know where it goes! So, we need some different quantities to describe the motion (displacement). There are two different quantities in the description of the physical events:

- Scalars
- Vectors

Quantities that can be specified completely by a number and/or unit and that have only magnitude are called "*scalars*". For example, distance (x), mass (m), time (t), volume (V), density (d), work (W), energy (E)…etc.

Quantities that behave like displacements are called "*Vectors*". For example, displacement (Δx), velocity (v), acceleration (a), force (F), moment (M), weight (G)…etc. (Vector means "*carrier*" in Latin. In Biology the term "vector" means an insect, animal or an agent that carries a cause of disease from one organism to another)

Vectors have both magnitude and direction, and obey certain rules of combination.

Graphically, a vector is represented by an arrow defining the direction. The length of the arrow defines the vector's magnitude. This is shown in Figure 1.
If we denote one end of the arrow by the origin O and the tip of the arrow by Q. Then the vector may be represented algebraically by OQ.

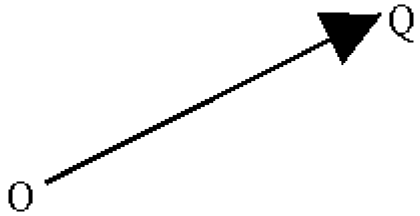


Figure 1

This is often simplified to just $\vec{Q}$ or $\bar{Q}$. The line and arrow above the Q are there to indicate that the symbol represents a vector. Another notation is boldface type as: **Q**.

Note, that since a direction is implied, $OQ \neq QO$. Even though their lengths are identical, their directions are exactly opposite, in fact **OQ = -QO**.

The magnitude of a vector is denoted by absolute value signs around the vector symbol: magnitude of **Q = |Q|**.

Two vectors are always coplanar. Three or more vectors are said to be coplanar if they are parallel to the same plane or lie in the same plane however their magnitudes can be different.

The operation of addition, subtraction and multiplication of ordinary algebra can be extended to vectors with some new definitions and a few new rules. There are two fundamental definitions.

#1 Two vectors, **A** and **B** are equal if they have the same magnitude and direction, regardless of whether they have the same initial points, as shown in Figure 2.

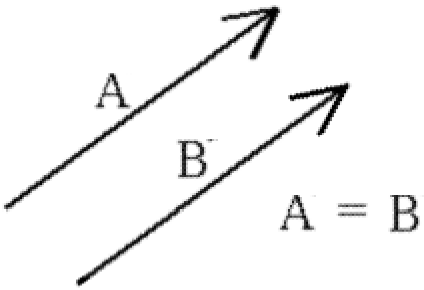


Figure 2

#2 A vector having the same magnitude as **A** but in the opposite direction to **A** is denoted by **-A** , as shown in Figure 3.

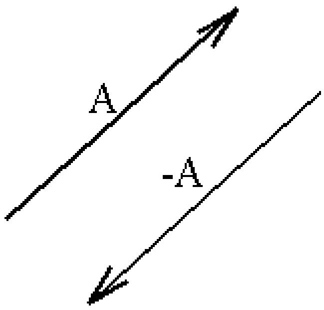


Figure 3

1. Vector Additions

1.a. Graphical Method

We can now define <u>vector addition</u>. The sum of two vectors, **A** and **B**, is a vector **C**, which is obtained by placing the initial point of **B** on the final point of **A**, and then drawing a line from the initial point of **A** to the final point of **B** , as illustrated in Figure 4. This is sometines referred to as the "Tip-to-Tail" method.

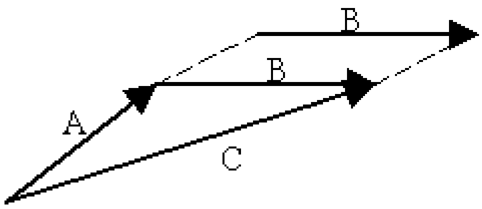


Figure 4

Vector subtraction is defined in the following way. The difference of two vectors, **A - B** , is a vector **C** that is, **C = A - B** or **C = A + (-B)**. Thus vector subtraction can be represented as a vector addition.

The graphical representation is shown in Figure 5. Inspection of the graphical representation shows that we place the initial point of the vector **-B** on the final point the vector **A** , and then draw a line from the initial point of **A** to the final point of **-B** to give the difference **C**.

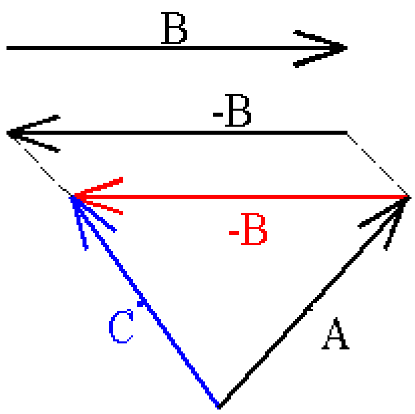


Figure 5

As we mentioned above, any quantity that has a magnitude but no direction associated with it is called a **"scalar"**. Speed, mass and temperature are scalars, for example. The product of a scalar, p say, times a vector **A** , is another vector, **B**, where **B** has the same direction as **A** but the magnitude is changed, that is, |**B**| = m|**A**|.

Many of the laws of ordinary algebra hold also for vector algebra. These laws are:

Commutative Law for Addition: A + B = B + A

Associative Law for Addition: A + (B + C) = (A + B) + C

The verification of the Associative law is shown in Figure 6.
If we add **A** and **B** we get a vector **E**. And similarly if **B** is added to **C**, we get **F** .
Now **D = E + C = A + F**.
Replacing **E** with (**A + B**) and **F** with (**B + C**), we get **(A +B) + C = A + (B + C)** and we see that the law is verified.
Stop now and make sure that you follow the above proof.

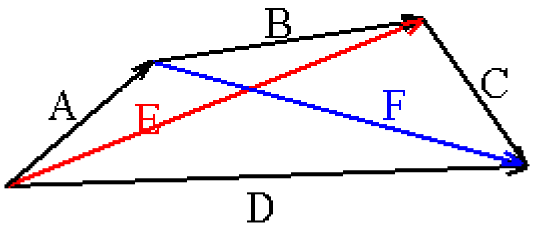


Figure 6

Commutative Law for Multiplication: p**A = A**p

Associative Law for Multiplication: (p + n)**A** = p**A** + n**A**, where p and n are two different scalars.

Distributive Law: p**(A + B)** = p**A** + p**B**

These laws allow the manipulation of vector quantities in much the same way as ordinary algebraic equations.

Vectors can be related to the basic coordinate systems which we use by the introduction of what we call "**unit vectors**." A unit vector is one, which has a magnitude of 1 and is often indicated by putting a hat (or circumflex) on top of the vector symbol, for example $\text{unit vector} = \hat{a}$. $|\hat{a}| = 1$. The quantity $\hat{a}$ is read as "a hat" or "a unit".

Let us consider the two-dimensional (or x, y) Cartesian Coordinate System, as shown in Figure 7.

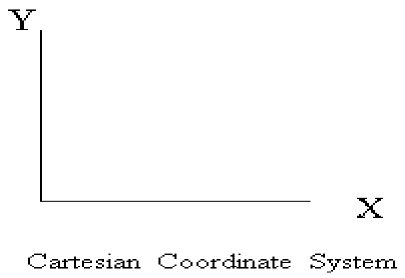


Figure 7

We can define a unit vector in the x-direction by $\hat{x}$ or it is sometimes denoted by $\hat{i}$.Similarly in the y-direction we use $\hat{y}$ or sometimes $\hat{j}$. Any two-dimensional vector can now be represented by employing multiples of the unit vectors, $\hat{x}$ and $\hat{y}$, as illustrated in Figure 8.

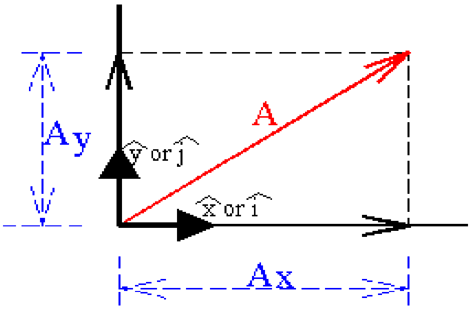


Figure 8

1.b. Analytical Method

The vector **A** can be algebraically represented by $\mathbf{A} = \mathbf{A_x} + \mathbf{A_y}$ where $\mathbf{A_x}$ and $\mathbf{A_y}$ are vectors in the x and y directions. If A_x and A_y are the magnitudes of $\mathbf{A_x}$ and $\mathbf{A_y}$, then $A_x\hat{x}$ and $A_y\hat{y}$ are the vector components of **A** in the x and y directions respectively.

The actual operation implied by this is shown in Figure 9.

Remember $\hat{x}$ (or $\hat{i}$) and $\hat{y}$ (or $\hat{j}$) have a magnitude of 1 so they do not alter the length of the vector, they only give it its direction.

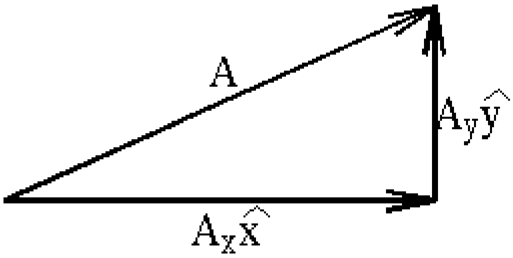


Figure 9

The breaking up of a vector into it's components, makes the determination of the length of the vector quite simple and straight forward.

Since $\mathbf{A} = A_x\hat{x} + A_y\hat{y}$, then using Pythagorus' Theorem $\mathbf{A} = \sqrt{A_x^2 + A_y^2}$.

For example

If $\mathbf{A} = 3\hat{x} + 4\hat{y}$

then $|\mathbf{A}| = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$.

The resolution of a vector into it's components can be used in the addition and subtraction of vectors. To illustrate this let us consider an example, what is the sum of the following three vectors?

A = $A_x\dot{x} + A_y\dot{y}$

B = $B_x\dot{x} + B_y\dot{y}$

C = $C_x\dot{x} + C_y\dot{y}$

By resolving each of these three vectors into their components we see that the result is Figure 11.

$D_x = A_x + B_x + C_x$

$D_y = A_y + B_y + C_y$

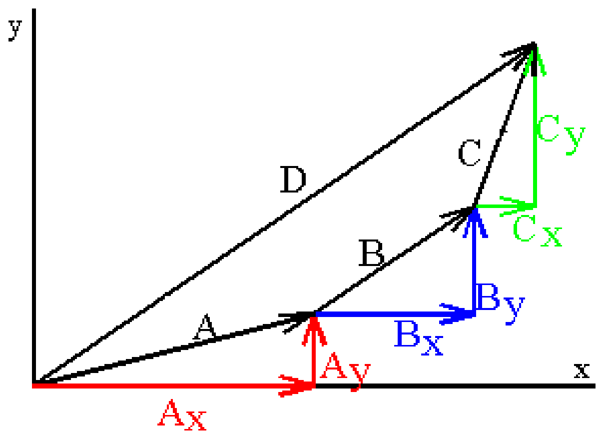


Figure 11

Very often in vector problems you will know the length, that is, the magnitude of the vector and you will also know the direction of the vector. From these you will need to calculate the Cartesian components, that is, the x and y components.

The situation is illustrated in Figure 12. Let us assume that the magnitude of **A** and the angle θ are given; what we wish to know is, what are A_x and A_y?

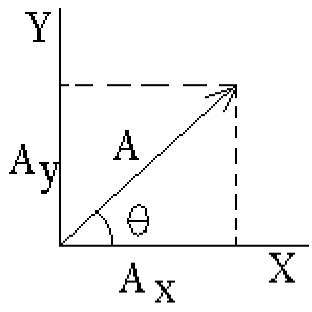


Figure 12

From elementary trigonometry we have, that
$\cos\theta = A_x/|\mathbf{A}|$ therefore $A_x = |\mathbf{A}|\cos\theta$, and similarly $A_y = |\mathbf{A}|\cos(90 - \theta) = |\mathbf{A}|\sin\theta$.

Polar Coordinate

Until now, we have discussed vectors in terms of a Cartesian, that is, an x-y coordinate system. Any of the vectors used in this frame of reference were directed along, or referred to, the coordinate axes. However there is another coordinate system, which is very often encountered and that is the **Polar Coordinate System.**

In Polar coordinates one specifies the length of the line and it's orientation with respect to some fixed line. In Figure 13, the position of the dot is specified by it's distance from the origin, that is r, and the position of the line is at some angle θ, from a fixed line as indicated. The quantities r and θ are known as the **Polar Coordinates** of the point.

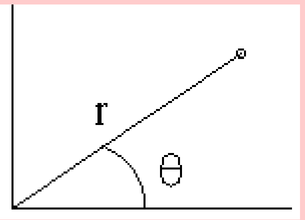


Figure 13

It is possible to define fundamental unit vectors in the Polar Coordinate system in much the same way as for Cartesian coordinates. We require that the unit vectors be perpendicular to one another, and that one unit vector be in the direction of increasing r, and that the other is in the direction of increasing θ.

In Figure 14, we have drawn these two unit vectors with the symbols $\hat{r}$ and $\hat{\theta}$.

It is clear that there must be a relation between these unit vectors and those of the Cartesian system.

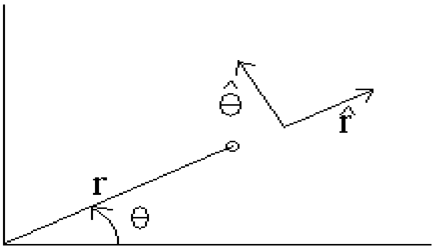
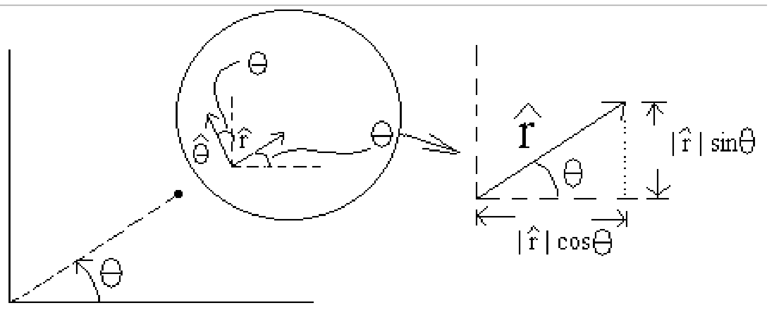


Figure 14

These relationships are given in Figure 15.



$$\hat{r} = \cos\theta\,\hat{x} + \sin\theta\,\hat{y}$$
$$\hat{\theta} = -\sin\theta\,\hat{x} + \cos\theta\,\hat{y}$$

Figure 15

2. Vector Multiplication

- Can we multiply vectors? When we add scalar quantities, the items being added must have the same dimensions, and the sum has the same. On the other hand, we can multiply scalar quantities of different dimensions, obtain a product having different dimension. Such as, x=v*t. Like scalars, vectors of different kind can be multiplied by one another. We have 3 kinds of multiplication:
    1. Multiplication of a vector by a scalar in such a way as to yield a vector,
    2. Multiplication of 2 vectors in such a way as to yield a new vector (Cross-X-product)

3. Multiplication of 2 vectors in such a way as to yield a Scalar (Scalar- . -product)

- The multiplication of two vectors is not uniquely defined, in the sense that there is a question as to whether the product will be a vector or not. For this reason there are two types of vector multiplication.

2.a. Scalar Product (or "dot product")

$$\mathbf{A}\cdot\mathbf{B} = |\mathbf{A}|\,|\mathbf{B}|\cos\theta$$

The scalar product of two vectors, **A** and **B** denoted by **A·B**, is defined as the product of the magnitudes of the vectors times the cosine of the angle between them, as illustrated in Figure 16.

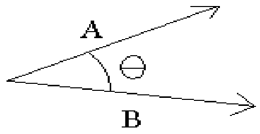


Figure 16

Note that the result of a dot product is a scalar, not a vector.

The rules for scalar products are given in the following list,

$$(1)\ \vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$$
$$(2)\ \vec{A}\cdot(\vec{B}+\vec{C}) = \vec{A}\cdot\vec{B} + \vec{A}\cdot\vec{C}$$
$$(3)\ m(\vec{A}\cdot\vec{B}) = (m\vec{A})\cdot\vec{B} = \vec{A}\cdot(m\vec{B}) = (\vec{A}\cdot\vec{B})\,m$$

And in particular we have $\hat{x}\cdot\hat{x} = \hat{y}\cdot\hat{y} = 1$, since the angle between a vector and itself is 0 and the cosine of 0 is 1.

Alternatively, we have $\hat{x}\cdot\hat{y} = 0$, since the angle between $\hat{x}$ and $\hat{y}$ is 90º and the cosine of 90º is 0. In general then, if **A·B** = 0 and neither the magnitude of **A** nor **B** is 0, then **A** and **B** must be perpendicular. The definition of the scalar product given earlier, required a knowledge of the magnitude of **A** and **B** , as well as the angle between the two vectors. If we are given the vectors in terms of a Cartesian representation, that is, in terms of $\hat{x}$ and $\hat{y}$, we can use the information to work out the scalar product, without having to determine the angle between the vectors.

If

$$\vec{A} = A_x\hat{x} + A_y\hat{y}$$
$$\vec{B} = B_x\hat{x} + B_y\hat{y}$$

, then

$$\vec{A}\cdot\vec{B} = (A_x\hat{x} + A_y\hat{y})\cdot(B_x\hat{x} + B_y\hat{y})$$
$$= A_x\hat{x}\cdot(B_x\hat{x} + B_y\hat{y}) + A_y\hat{y}\cdot(B_x\hat{x} + B_y\hat{y})$$
$$= A_xB_x + A_yB_y$$.

Because the other terms involved, $\hat{x}\cdot\hat{y} = 0$, as we saw earlier.

Let us do an example. Consider two vectors, $\vec{A} = 2\hat{x} + 2\hat{y}$ and $\vec{B} = 6\hat{x} - 3\hat{y}$.
Now what is the angle between these two vectors?
From the definition of scalar products we have

$$\vec{A}\cdot\vec{B} = |\vec{A}|\,|\vec{B}|\cos\theta$$
$$\therefore \cos\theta = \frac{\vec{A}\cdot\vec{B}}{|\vec{A}|\,|\vec{B}|}$$.

But

$$\vec{A}\cdot\vec{B} = 2x6 - 3x2 = 6$$
$$|\vec{A}| = \sqrt{2^2 + 2^2} = \sqrt{8} = 2.83$$
$$|\vec{B}| = \sqrt{6^2 + (-3)^2} = \sqrt{45} = 6.71$$
$$\therefore \cos\theta = \frac{6}{2.83 \times 6.71} = 0.316$$
$$\therefore \theta = 71^{0}\,36'$$

2.b. Vector Product (or Cross Product)

The vector product or the cross product multiplies two vectors in such a way that the resultant is a new vector.

If **A**=$A_xi+A_yj+A_zk$ and **B**=$B_xi+B_yj+B_zk$, then let **C** = $C_x\, i + C_y\, j + C_y\, k$ be the result of this multiplication.

Let ***A***=*i+2j+3k;* **B**=*2i-3j+k*. Then the cross product of **A** and **B** is
$\vec{C} = \vec{A}\times\vec{B} = |\vec{A}||\vec{B}|\sin\theta\,\hat{e}_C$ where $\hat{e}_C$ is the unit vector in the direction of vector **C**, and it can be found using right-hand rule.

(Right Hand Rule: Curve your fingers of the right hand from *A to* B (so that A and B form a plane) the thumb will point in the direction of C. Note: A x B is not the same as B x A (while A · B is equal to B · A)

The vector **C** can also be found by

$$\vec{C} = \begin{bmatrix} i & j & k \\ 1 & 2 & 3 \\ 2 & -3 & 1 \end{bmatrix} = i(2*1-3*(-3)) - j(1*1-3*2) + k(-3*1-2*2)$$

$$\vec{C} = 11i + 5j - 7k$$

The cross product among unit vectors:

$\boldsymbol{i} \times \boldsymbol{i} = \vec{0}$; **included angle is 0**

$\boldsymbol{i} \times \boldsymbol{j} = \boldsymbol{k}$; **included angle is 90°**

$\boldsymbol{j} \times \boldsymbol{i} = -\boldsymbol{k}$; $A \times B \neq B \times A$

- This concludes our survey of the elementary properties of Vectors. We have concentrated on fundamentals and have not restricted ourselves to the discussion of vectors in just two dimensions. Nevertheless, sound grasp of the ideas presented in this lecture are absolutely essential for further progress in vector analysis.

Examples and Problems

Question 2.1:
Consider the two vectors

with angle between them. Find the vector |**S**|, defined to be the cross product of **A** and **B**.

Solution 2.1:

The magnitudes of the vectors **A** and **B** are computed as follows:

$$A = \sqrt{(4)^2 + (-1)^2 + (0)^2} = 4.12$$

and

$$B = \sqrt{(13)^2 + (8)^2 + (0)^2} = 8.10$$

Finding the *magnitude* of the vector |**S**| is then easy:

$S = AB\sin\theta = (4.12)(8.10)\sin(94.8^\circ) = 33.3$
Remember the symbols for denoting a vector pointing directly *out of* or directly *into* the screen (or piece of paper, etc.)

Question 2.2:
If three vectors are given as

$\vec{A} = i + j + k, \quad \vec{B} = 2i - j + 3k \quad and \quad \vec{C} = i + 2j - k,$

then find the results of following equations given below:

a) $\overrightarrow{(A+B)}.\vec{C} \quad and \quad \vec{C}.\overrightarrow{(A+B)}$ (This is the commutative relation! ☺)
b) $\vec{A}\times(\vec{B}+\vec{C}) \quad and \quad (\vec{B}+\vec{C})\times\vec{A}$ (Cross products are not commutative! ☹)
c) $(\vec{A}\times\vec{B}).\vec{C} \quad and \quad \vec{A}.(\vec{B}\times\vec{C})$
d) $\vec{A}\times(\vec{B}\times\vec{C})$

Solution 2.2:

a) $\overrightarrow{(A+B)}.\vec{C} \quad and \quad \vec{C}.\overrightarrow{(A+B)}$

$\overrightarrow{(A+B)} = (i+j+k)+(2i-j+3k) = 3i+4k$

$\overrightarrow{(A+B)}.\vec{C} = (3i+4k).(i+2j-k) = 3+0-4 = -1$

this result is same for the equation $\vec{C}.\overrightarrow{(A+B)} = -1$.

b) $\vec{A}\times(\vec{B}+\vec{C}) \quad and \quad (\vec{B}+\vec{C})\times\vec{A}$

$(\vec{B}+\vec{C}) = (2i-j+3k)+(i+2j-k) = 3i+j+2k$

$$\vec{A}\times(\vec{B}+\vec{C}) = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 3 & 1 & 2 \end{vmatrix} = i + j - 2k \quad and$$

$$(\vec{B}+\vec{C})\times\vec{A} = \begin{vmatrix} i & j & k \\ 3 & 1 & 2 \\ 1 & 1 & 1 \end{vmatrix} = -i - j + 2k. Then \quad \vec{A}\times(\vec{B}+\vec{C}) = -(\vec{B}+\vec{C})\times\vec{A}$$

c) $(\vec{A}\times\vec{B}).\vec{C} \quad and \quad \vec{A}.(\vec{B}\times\vec{C})$

$\vec{A}\times\vec{B} = 4i - j - 3k \quad and \quad (\vec{A}\times\vec{B}).\vec{C} = 5$

$\vec{B}\times\vec{C} = -5i + 5j + 5k \quad and \quad \vec{A}.(\vec{B}\times\vec{C}) = 5$ ☺

d) This will be done by students! ☺

Question 2.3

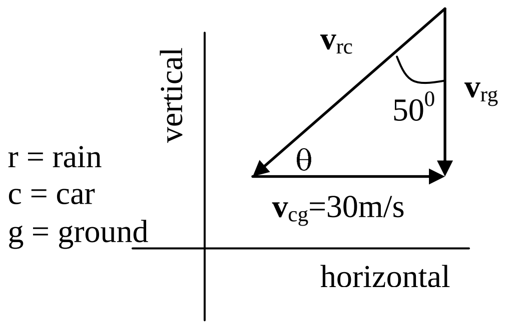
Suppose that an airplane is aimed due east with an airspeed of 200mi/h while a wind is blowing due north at 80mi/h. Find the velocity of the plane relative to the observer on the ground?

Solution 2.3
The velocity of the plane relative to the air is, v_{pa}^2, 200mi/h due east and the velocity of the air relative to the earth is, v_{ae}^2, 80mi/h due north. Then, it is seen that the velocity vectors are at right angles. Since they are at right angles, the velocity of the plane relative to the earth, v_{pe}, is found as:
$v_{pe} = (v_{pa}^2 + v_{ae}^2)^{1/2}$, then $v_{pe} = (200^2 + 80^2)^{1/2}$=215mi/h
the angle that the plane flies in the direction of N of E is

$$\tan\theta = \frac{80}{200} = 0.40, \quad then \quad \theta = 21.8^\circ .$$

Question 2.4
In a rainy day, the raindrops fall vertically downward to the ground. A physics professor driving carefully a car horizontally at 30m/s (!) observes that the rain forms streak lines on the side windows that make an angle of 50^0 with the vertical. The professor wonders what the speed of the raindrops relative to the ground?

Solution 2.4:
As it is seen, the streak lines represent the direction of the raindrops relative to the car. It is also given that raindrops fall vertically to the ground. Then, it can be drawn as seen in figure. Then $\mathbf{v}_{rg}=\mathbf{v}_{rc}+\mathbf{v}_{cg}$ is obtained. It is known that $\mathbf{v}_{cg}$=30m/s and $\theta=40^0$. So that we can find the speed of the raindrops relative to the ground:



$v_{rg} = v_{cg} \tan\theta$ = (30m/s) (0.839) = 25.2m/s